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FACTORS IN THE LEARNING BEHAVIOR OF THE ALBINO RAT

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The table of intercorrelations published by Dr. R. L. Thorndike in the *Genetic Psychology Monographs*, 1935, **17**, No. 1, has been reanalyzed into ten factors. Rotation of the reference vectors resulted in a configuration which, for all practical purposes, may be said to show a simple structure and (except for one variable on one factor) a positive manifold. Five of the factors can be interpreted with some confidence, one only with considerable caution; three factors are specific to the apparatus employed, and for the one remaining factor, no interpretation is attempted; it appears to be a residual plane.

For some years data obtained by experimentation on animal learning have been analyzed statistically. In the beginning of this period such investigators as Davis and Tolman (**3**) and others hoped to find a single most satisfactory criterion of learning, i.e., a criterion having the maximal reliability and validity. Later the assumption of a unitary learning factor was attacked and shown to be untenable and it was demonstrated that several unrelated factors were involved [e.g., Lashley (**13**), Leeper (**14**), Commins, McNemar and Stone (**2**)]. With reference to this complexity, Tomilin and Stone (**25**) wrote in 1934:

> We look upon this discovery as one of capital importance because now-a-days many problems in animal psychology are attacked by laboratory methods masquerading under the same name, but differing from each other in many essential details. So long as we lack accurate information as to the community of function of these diverse techniques of experimentation, just so long shall we continue to grope in the dark while seeking plausible explanations of the divergent results obtained on fundamental problems.

There is still another (though related) reason why learning, particularly animal learning, needs a thorough analysis. Traditionally, descriptions of such learning have confined themselves to terms involving laboratory apparatus employed; thus, we have maze behavior or rather behavior in particular mazes, and the relationship of this behavior to that on a puzzle box or to that shown on a delayed response apparatus. Such descriptions are, in the last analysis, chaotic and otherwise unsatisfactory. Experimentation is never-ending and predictions of future behavior of the animals in other situations are unreliable.

It is the purpose of this paper to investigate some measures of animal learning behavior and to isolate, if possible, the number of independent variables that must be dealt with in problems of this kind. For this purpose, attention was turned to a study by R. L. Thorndike which appeared in the *Genetic Psychology Monographs*, 1935, **17**, No. 1. It contained (among other items) a table of intercorrelations between scores on performances on a latch box, a number of mazes and other problems in learning. Thorndike had extracted three centroid factors from this table, but had stopped at this point because of a faulty criterion.* Furthermore, no attempt was made: (1) to determine whether a simple structure† would fit the configuration of vectors, or (2) to rotate the axes in any other manner. It was thought that considerably more information could be obtained from this table than had been extracted from it. Hence, a fresh attack was made on it to answer, if possible, such questions as:

1. Can animal behavior be parsimoniously described in terms of a limited number of abilities; i.e., can the above-mentioned table of intercorrelations of behavior measures be duplicated (allowing for small observational errors) by employing a substantially smaller number of factors than was made use of in the original composition of the table?

2. Will such abilities be found to be independent of one another; i.e., are the primary factors obtained orthogonal (uncorrelated)?

3. Do any of the abilities found render "good" performance (see page 182) on any of the tasks in question more difficult; that is, are any of these factor loadings negative, so that they cannot be enclosed within a positive manifold?

4. What are the verbal interpretations of the several abilities employed by the animals and methodologically isolated?

Partial Résumé of Thorndike's Report. In order that the reader may understand the results of the present investigation from this report, I shall give a résumé of the relevant material from Thorndike's report.

There are thirty-two variables representing different kinds of scores. They are presented in the temporal order of performance of the various tasks.

* The criterion was that the residuals from the whole table should form a distribution which approximated normality and should have for a standard deviation the standard error of the correlation coefficient derived on the assumption that the true correlation between the variables is zero.

† By simple structure is meant a selection of defining axes which maximizes the number of vanishing factor loadings, i.e., maximizes the number of zero entries in matrix F_f.

(1) Number of turns made on a revolving wheel activity cage.

Warner-Warden Mazes, *A* and *B* (alley mazes).

Maze *A* (see sketch, Plate I. Adapted from Thorndike's article, p. 24).

(2) "*A* errors" — entering a blind alley while progressing toward a goal box.
(3) "*B* errors" — entering a blind alley while proceeding in the reverse direction.
(4) "*C* errors" — retracing the true pathway.
(5) Total time (from entrance box to goal box).

Maze *B* (see sketch, Plate I).

(6) "*A* errors."
(7) "*B* errors."
(8) "*C* errors."
(9) Total time.

Elevated Mazes *C* and *D*.

Maze *C* (see sketch, Plate I).

(10) "*A* errors."
(11) "*B* errors."
(12) "*C* errors."
(13) Time to start running.
(14) Time to run.

Maze *D* (see sketch, Plate I).

(15) "*A* errors."
(16) "*B* errors."
(17) "*C* errors."
(18) Time to start running.
(19) Time to run.

Jenkins Circular Problem Box (see sketch, Plate I. Adapted from F. L. Riess, Limits of learning in the white rat and the guinea pig, *Genet. Psychol. Monog.*, 1934, **15**, p. 313).

(20) Number of quadrants entered before reaching food compartment door sprung by stepping on floor plates.

(21) Perfect trials (a more or less subjective estimate of the excellence of the animal's performance).

(22) Time to start from entrance box.
(23) Time to run.

Latch Box Problem (see sketch, Plate I).

(24) Time to reach food compartment.

Conditioned Response (see sketch, Plate I. Adapted from L. N. Warner, Association span of the white rat, *J. Genet. Psychol.*, 1932, **41**, p. 64).

Animal required to jump across a fence to avoid shock which follows ten seconds after a stimulus, which, in variables [25] and [26], was a buzzer.

(25) Score — number of crossings (of fence) following buzzer.

(26) Number of crossings in the intervals between trials.

(27) Same as twenty-five except a light was substituted as the conditioned stimulus.

(28) Number of crossings in the intervals between trials.

Columbia Obstruction Apparatus (see sketch, Plate I. Adapted from T. R. Jenkins, L. N. Warner, and C. J. Warden, Standard apparatus for the study of animal motivation, *J. Comp. Psychol.*, 1926, **6**, p. 366).

Animal motivated by one-day hunger was required to cross electric grid to reach food compartment.

(29) Number of approaches to the grid.

(30) Number of contacts with the grid (resulting in shock).

(31) Number of complete crossings of the grid.

Feeding on Preliminaries.

(32) The ratio of the time actually spent in eating to the total time available for this purpose, obtained on the preliminary training on Maze A, the Jenkins Box and the obstruction apparatus.

The scores of these variables were correlated and the direction of the variable chosen in such a manner as to call all "good" scores the same sign. Whatever a "good" score was to be was arbitrarily defined, and the following were chosen as "good" scores:

Revolving Wheel — large number of turns.
Mazes — few errors or short time.
Problem boxes — short time, few quadrants entered, and many perfect trials.
Conditioned Response — many crossings of the fence.
Columbia Obstruction Box — many approaches, contacts or crossings.

The upper right portion of Table I (above the diagonal) shows the matrix of raw intercorrelations obtained in this manner and found on pages 42 and 43 of Thorndike's monograph.

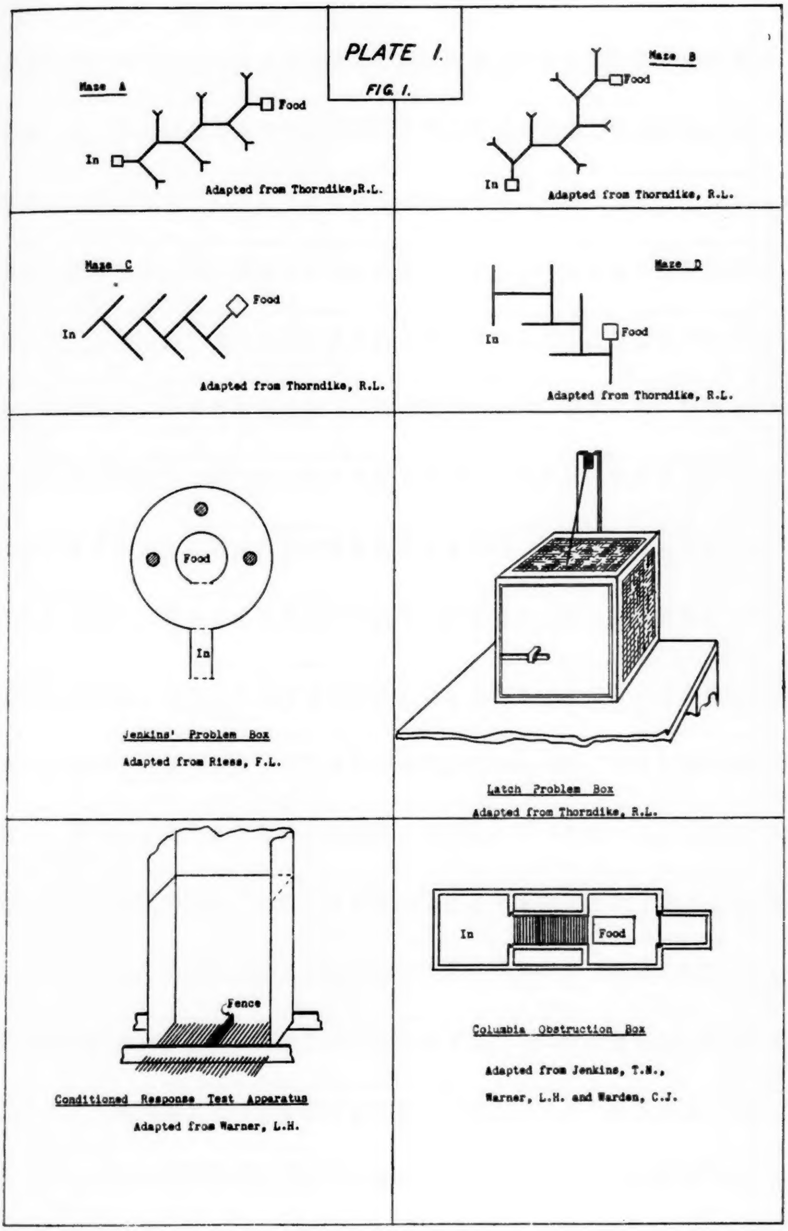
PLATE I.
FIG. 1.
Maze A
In
Food
Adapted from Thorndike,R.L.
Maze B
Food
In
Adapted from Thorndike, R.L.
Maze C
In
Food
Adapted from Thorndike, R.L.
Maze D
In
Food
Adapted from Thorndike, R.L.
Food
In
Jenkins' Problem Box
Adapted from Riess, F.L.
Latch Problem Box
Adapted from Thorndike, R.L.
Fence
Conditioned Response Test Apparatus
Adapted from Warner, L.H.
In
Food
Columbia Obstruction Box
Adapted from Jenkins, T.N.,
Warner, L.H. and Warden, C.J.

TABLE I

Entries below the diagonal: the matrix R obtained by multiplying the factorial matrix F_c by F'_c;
Entries above the diagonal: the empirically obtained correlations as reported by Thorndike

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1		.27	.25	.23	.36	.13	.07	.12	.19	.26	.25	.23	.25	.25	.03	.06
2	.26		.79	.88	.82	.39	.43	.42	.29	.53	.32	.57	.46	.57	.32	.39
3	.25	.82		.87	.81	.36	.46	.42	.33	.56	.40	.61	.46	.56	.16	.37
4	.24	.87	.87		.84	.26	.47	.38	.28	.56	.34	.61	.48	.58	.14	.32
5	.29	.82	.81	.83		.29	.39	.36	.41	48.	.39	.61	.59	.69	.09	.26
6	.15	.36	.33	.27	.29		.74	.85	.74	.20	.04	.22	.11	.10	.34	.35
7	.10	.43	.49	.43	.38	.79		.91	.79	.18	—.04	.22	.18	.09	.26	.37
8	.11	.42	.43	.38	.37	.86	.91		.82	.13	—.03	.23	.13	.07	.19	.32
9	.17	.30	.32	.28	.39	.75	.78	.81		.12	—.01	.22	.27	.17	.15	.16
10	.26	.55	.56	.56	.48	.19	.19	.15	.09		.50	.72	.45	.58	.40	.26
11	.21	.34	.37	.35	.38	.05	—.03	—.05	.03	.58		.68	.53	.68	.40	.26
12	.27	.58	.60	.60	.62	.22	.21	.21	.20	.65	.68		.58	.81	.23	.18
13	.25	.44	.49	.49	.62	.13	.17	.13	.32	.43	.52	.62		.74	.12	.08
14	.27	.54	.57	.58	.69	.09	.10	.07	.20	.57	.67	.79	.74		.17	.07
15	.04	.27	.19	.18	.09	.33	.23	.21	.16	.40	.36	.32	.12	.19		.38
16	.06	.37	.40	.32	.26	.36	.35	.29	.16	.26	.17	.19	.04	.12	.40	
17	.04	.35	.35	.28	.20	.38	.36	.29	.17	.26	.15	.16	.01	.07	.50	.80
18	.23	.24	.25	.23	.39	.33	.33	.27	.58	.17	.23	.23	.49	.37	.20	.16
19	.16	.46	.41	.39	.38	.39	.33	.30	.29	.42	.37	.42	.30	.37	.51	.50
20	—.21	.12	.02	.02	.00	.22	.14	.18	.10	—.02	—.04	—.14	—.17	—.20	.17	.18
21	—.22	.00	—.09	—.08	—.15	.25	.17	.23	.11	—.03	—.06	—.15	—.26	—.28	.17	—.01
22	.32	.46	.30	.31	.50	.43	.28	.33	.52	.23	.22	.32	.43	.39	.30	.23
23	.17	.49	.28	.30	.43	.48	.33	.40	.48	.15	.07	.14	.23	.17	.31	.26
24	.08	.14	.23	.16	.16	.24	.30	.20	.26	.11	.07	.03	.14	.06	.22	.45
25	.32	.16	.17	.12	.07	.21	.09	.11	.01	.27	.23	.12	—.01	.00	.11	.10
26	.28	.26	.20	.16	.21	.17	—.01	.10	—.07	.16	.10	.14	.00	.08	—.02	.15
27	.21	—.02	.01	—.06	—.08	.14	—.03	.00	—.07	.16	.22	.02	—.10	—.08	.10	.15
28	.39	.20	.20	.14	.16	.33	.18	.25	.14	.18	.12	.10	.02	.02	.03	.15
29	—.04	.05	.05	—.01	.11	.27	.21	.28	.23	—.07	—.10	.05	.14	.08	—.01	.05
30	—.06	.22	.24	.20	.15	.14	.16	.17	—.02	.11	—.03	.11	.09	.08	.10	.13
31	—.15	.14	.16	.12	.09	.18	.20	.17	.11	.12	.04	.10	.16	.10	.24	.13
32	.36	.35	.30	.32	.33	.18	.15	.11	.21	.35	.28	.36	.37	.39	.30	.16

TABLE 1 (continuel)

	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
1	—.01	.24	.21	—.24	—.22	.34	.21	.08	.34	.33	.19	.37	—.07	—.02	—.15	.38
2	.36	.28	.43	.13	—.02	.43	.44	.14	.20	.27	.01	.17	.06	.17	.11	.40
3	.35	.26	.41	.04	—.07	.32	.29	.25	.13	.22	.01	.23	—.01	.29	.16	.28
4	.29	.24	.39	.03	—.10	.31	.29	.14	.10	.16	—.04	.14	.02	.15	.11	.36
5	.20	.40	.35	.00	—.13	.52	.45	.15	.05	.20	—.09	.15	.10	.17	.10	.34
6	.32	.31	.40	.27	.24	.40	.48	.26	.25	.15	.14	.33	.27	.12	.19	.19
7	.33	.33	.34	.13	.15	.29	.33	.26	.09	.00	—.03	.19	.25	.14	.20	.15
8	.29	.23	.31	.15	.22	.35	.41	.21	.15	.09	.03	.22	.27	.14	.18	.07
9	.21	.60	.25	.06	.11	.55	.49	.26	—.04	—.06	—.08	.15	.26	—.02	.15	.18
10	.30	.18	.36	—.02	—.04	.22	.16	.08	.24	.12	.19	.15	—.08	.09	.16	.32
11	.05	.21	.43	—.07	—.02	.26	.08	.02	.22	.11	.22	.12	—.13	.00	.07	.29
12	.20	.21	.36	—.17	—.16	.35	.16	.06	.10	.09	.05	.14	.03	.13	.10	.35
13	.05	.54	.33	—.14	—.30	.41	.21	.13	.10	.00	—.10	—.06	.20	.10	.11	.33
14	.10	.40	.37	—.17	—.26	.39	.18	.09	.02	.07	—.07	.05	.03	.12	.09	.38
15	.55	.29	.54	.20	.20	.26	.28	.22	.11	—.01	.07	.04	.01	.17	.18	.31
16	.85	.10	.50	.17	—.08	.23	.25	.44	.13	.13	.18	.13	.06	.10	.14	.14
17		.22	.52	.17	.03	.32	.32	.48	.07	.09	.15	.08	.04	.17	.18	.13
18	.19		.34	.03	—.12	.63	.47	.39	.00	—.13	—.06	.10	.09	—.14	.09	.39
19	.55	.34		.17	.01	.42	.38	.33	.12	.11	—.02	.10	.11	.08	.16	.42
20	.20	.01	.10		.51	.02	.39	.07	.22	.13	.25	.19	.06	.05	.13	—.13
21	.01	—.11	—.06	.48		—.05	.24	—.07	.29	.13	.26	.17	—.02	.04	.11	—.21
22	.27	.61	.45	.09	—.05		.83	.21	.05	.15	.00	.19	.16	.08	.14	.46
23	.31	.49	.41	.36	.22	.77		.27	.17	.27	.10	.29	.23	.16	.22	.31
24	.48	.36	.31	.08	—.07	.23	.22		.05	—.04	.10	.11	.03	.14	.22	.20
25	.05	.00	.08	.22	.27	.06	.17	.08		.54	.77	.71	—.05	.11	.01	.12
26	.10	—.13	.07	.13	.08	.16	.25	—.03	.54		.45	.64	.27	.26	.04	.07
27	.10	—.04	.04	.26	.28	—.02	.09	.09	.75	.47		.64	—.08	.00	—.06	—.02
28	.11	.06	.10	.15	.15	.19	.27	.11	.74	.61	.65		.14	.16	—.01	.19
29	.06	.08	.07	.02	—.03	.22	.26	.08	—.04	.22	—.08	.12		.49	.45	.13
30	.16	—.10	.12	.06	.02	.04	.15	.09	.11	.27	.01	.16	.48		.53	.07
31	.19	.11	.20	.13	.10	.13	.23	.19	.02	.04	—.05	.00	.42	.52		.21
32	.21	.38	.36	—.15	—.21	.47	.33	.20	.12	.08	—.01	.15	.07	.11	.14	

TABLE 2

Matrix F_c — The centroid factor loadings, the communalities and the reliabilities of the 32 variables

Var.	Factor											
	I	II	III	IV	V	VI	VII	VIII	IX	X	h^2	r_a
1	.338	.235	–.179	–.351	.104	.191	–.186	–.235	.067	–.182	.499	.98
2	.759	.303	.154	.215	.183	.262	.144	.048	–.193	–.083	.907	.81
3	.729	.313	.113	.277	.281	.165	.106	–.136	–.037	.131	.873	.69
4	.683	.403	.149	.269	.279	.218	.116	–.076	–.159	.095	.902	.88
5	.713.	.458	.137	.038	.289	.167	.177	.134	.027	.088	.908	.84
6	.656	–.401	.247	–.071	.220	–.201	–.267	.059	.020	–.141	.841	.77
7	.611	–.292	.409	.038	.399	–.239	–.224	–.126	–.069	–.080	.921	.79
8	.609	–.362	.354	.007	.470	–.219	–.249	.038	–.052	–.073	.968	.81
9	.563	–.148	.417	–.325	.284	–.303	–.271	.139	.022	.092	.893	.91
10	.579	.349	–.193	.286	–.105	.006	–.160	–.107	–.077	–.032	.631	.47
11	.462	.439	–.322	.217	–.280	–.091	–.231	.103	.176	.071	.744	.53
12	.607	.524	–.195	.259	.046	–.168	–.129	.080	.097	–.095	.820	.49
13	.515	.548	–.087	–.128	.043	–.249	.068	.111	.102	.204	.722	.83
14	.552	.665	–.162	.084	.021	–.184	.028	.141	.186	.046	.872	.47
15	.468	–.071	.066	.225	–.487	–.188	–.163	–.080	–.149	–.194	.644	.67
16	.519	–.207	.286	.269	–.257	.147	.122	–.281	.342	–.078	.771	.54
17	.526	–.247	.336	.261	–.367	.079	.137	–.319	.263	–.146	.871	?
18	.494	.179	.265	–.488	–.189	–.193	–.063	.048	.062	.265	.738	.92
19	.623	.066	.162	.107	–.296	–.088	–.009	–.082	.052	–.146	.556	.55
20	.200	–.442	.083	.151	–.185	.221	.013	.351	–.131	.165	.516	.73
21	.086	–.487	–.022	.200	–.083	.108	–.221	.361	–.290	.122	.582	.41
22	.642	.132	.191	–.476	–.180	–.026	.100	.227	–.027	–.209	.832	.94
23	.649	–.168	.199	–.361	–.183	.131	.193	.351	–.215	–.126	.893	.90
24	.367	–.144	.219	–.101	–.215	–.039	.109	–.305	.179	.238	.455	.87
25	.391	–.294	–.579	–.092	–.002	.384	–.283	–.116	–.069	.126	.845	.87
26	.350	–.219	–.480	–.057	.177	.375	.143	.046	.126	–.243	.674	.90
27	.250	–.364	–.520	–.046	–.141	.380	–.295	–.038	.136	.146	.760	.90
28	.442	–.321	–.462	–.236	.154	.355	–.153	–.107	.079	–.036	.760	.89
29	.243	–.269	–.148	–.135	.218	–.394	.447	.116	.077	–.122	.608	.73
30	.292	–.223	–.273	.166	.160	–.219	.514	–.126	–.176	–.044	.624	.84
31	.309	–.217	–.129	.108	–.086	–.404	.412	–.044	–.226	.136	.583	?
32	.463	.279	–.041	–.261	–.164	–.096	.021	–.231	–.149	–.158	.499	.79

h^2 = communalities

r_a = split-half reliabilities augmented by the Spearman-Brown formula for tests twice as long

As previously stated, three centroid factors were defined by the loadings of this table, and an attempt was made to interpret them.

Further Treatment of the Data. In the present study the original table of intercorrelations was factored into ten significant centroid factors shown in Matrix F_c, Table 2. The significance of the number of factors employed is based upon certain criteria developed in Professor Thurstone's laboratory and not yet published. The table also contains the communality of each variable, h^2, i.e., the sum of the squares of the factor loadings for each variable. A comparison of these with the reliability of each measure as reported by Thorndike, which is also given, indicates the extent to which the present study accounts for all the abilities involved in the performance on the various tasks. Thus, as would be expected, the maze behavior is fairly well accounted for. The eighteen measurements enable us to represent maze-performance in a quite comprehensive manner. On the other hand, the performance on the revolving wheel activity cage shows a large discrepancy between the communality and the reliability. Here, then, is exhibited a complex of abilities which, though capable of reliable measurement, has comparatively little in common with the abilities made use of in the performance of the other tasks represented in this study and hence escapes identification.

A comparison of the reported reliabilities and the obtained communalities (see Table 2) indicates that exclusive of the mazes the reliabilities exceed the communalities. But the maze measurements often show a reversal of this condition. This reversal must be interpreted to mean that, contrary to the expectation of Thorndike, these reported reliabilities are too low. This statement is strengthened by an examination of the table of intercorrelations (see Table 1) in which, in twelve out of thirteen cases, the variables in question show one or more raw correlations with some other variable which exceed the reported augmented reliability. In the thirteenth case another intercorrelation equals the reported augmented reliability. On the other hand, the reversal might conceivably mean that the several measurements of the same maze, by being almost duplicates of one another, would give rise to common factors in the error space. In that case the communalities might be too high.

To indicate the extent to which the factorial Matrix F_c, found in Table 2, accounts for the original Matrix R (Table 1 *above* the diagonal), the matrix multiplication FF' has been carried out and the resultant values entered in Table 1, *below* the diagonal. The discrepancies between corresponding entries constitute the tenth factor residuals.

Rotation of Axes. After extracting the ten centroid factors

shown in Table 2, an attempt was made to rotate the reference axes to new positions in which they would conform to the principle of simple structure and positive manifold. Several methods of rotations were made use of; among them, the last one employed, was the method of extended vectors, a method recently devised by Thurstone (**22**). The final position of the reference axes is such that the cosines of the angles made by these new reference axes with the respective centroid reference vectors are those appearing in Table 3. The new reference vectors are tentatively denoted by capital letters, A, B, etc.

Thus Factor A, defining one of the hyperplanes, is selected such that it makes an angle of $+16^\circ\ 44'$ ($\cos^{-1}$ .288) with the first centroid axis, $\cos^{-1}$ .082 with the second, etc.

This new set of reference axes does not define a strictly orthogonal frame. The cosines of the angles which these axes make with one another are given in Table 4, below the diagonal entries. Thirty-one of these (forty-five) cosines indicate that the corresponding angles between reference vectors are more than 87° ($\cos^{-1} = .0523$). Of the six cosines which exceed $\pm.150$, four are descriptive of angles which Factor D makes with the other axes; this question will be considered further under the discussion of this axis. It cannot be stated whether or not one or more degrees of freedom would permit this structure to be described in a completely orthogonal frame, but since the maximum departure from orthogonality is $18^\circ\ 40'$, as indicated by a cosine of —.320, it may be said that for purposes of interpretation the factors are orthogonal. The cosines of the angles between the primaries themselves are found in Table 4, above the diagonal entries. This matrix is designated TT' and $TT' = D(\lambda\lambda')^{-1}D$.

Table 3

Matrix λ

Cosines of the angles subtended by the final and the centroid axes

Centroid Axes	Final									
	A	B	C	D	E	F	G	H	J	K
I	.288	.279	.210	.270	.322	.124	.129	.360	.141	.166
II	.082	-.218	-.365	.156	.132	-.283	-.331	.521	-.221	-.048
III	-.163	-.694	.189	.235	.071	-.402	-.044	-.195	.314	.024
IV	.276	-.245	.021	.094	-.680	.115	.299	.514	.202	-.342
V	-.546	.058	.582	.409	-.248	.149	-.287	.211	-.334	.130
VI	-.343	.342	-.427	.253	-.240	-.405	.140	.061	.193	.279
VII	-.309	-.288	-.469	.055	.170	.721	-.024	.036	.268	.212
VIII	-.082	-.330	.054	-.362	.167	-.056	.659	.358	-.332	.474
IX	-.173	.124	.184	-.672	.190	-.132	-.210	.284	.676	.140
X	-.509	.077	-.093	.150	.446	-.057	.449	.196	-.063	-.685

The factor loadings on the new reference axes are given in Table 5. It can be seen that they are, with the exception of variables [1] and [32], on Factor G, enclosed within a positive manifold. Other negative factor loadings are so small that they may easily be attributed to chance errors.

Table 4

Entries above diagonal: matrix $TT' = D(\lambda\lambda')^{-1}D$, cosines of angles between the primary factors established

Entries below diagonal: matrix $\lambda\lambda'$, cosines of the angles subtended by the reference vectors of the primary planes with each other

Vector	A	B	C	D	E	F	G	H	J	K
A	1.000	–.015	.058	.259	.259	–.006	.059	.003	.091	.065
B	.014	1.000	–.009	.025	–.044	–.009	.106	.020	.042	.043
C	–.010	.022	1.000	.100	.113	.046	.138	–.021	.160	.009
D	–.179	–.003	–.007	1.000	.261	–.008	.194	–.021	.367	.255
E	–.204	.036	–.101	–.214	1.000	–.016	–.045	.056	.065	.041
F	.001	.007	–.045	.004	.019	1.000	–.005	.000	.026	–.041
G	–.030	–.103	–.115	–.112	.095	.010	1.000	–.250	.213	.137
H	–.005	–.049	–.007	–.006	–.038	–.001	.248	1.000	–.035	–.011
J	.003	–.022	–.116	–.320	.028	–.025	–.135	–.010	1.000	.086
K	–.001	–.026	.022	–.222	.018	.038	–.094	–.018	.020	1.000

Psychological Interpretations of the Factors. In order that a factor may be identified and interpreted within a study such as this, the following conditions must be satisfied:

1. It must possess a number of loadings of sufficient magnitude to identify it unequivocally. Thurstone (**21**) uses loadings of .40 or larger.

2. The loadings should be derived from tasks of sufficient heterogeneity that the investigator may identify the common psychological element as distinct from the mode of measurement. Thus, several factors in the present study, while easily identified as primary factors, cannot be described in terms of psychological principles because they are represented only by performance on one kind of apparatus. For such factors further investigation should be made in order that the loadings obtained on the one task already tested may be compared with loadings on a different task. Such other tasks, though they employ the same ability, should differ sufficiently in mode of measurement (that is, apparatus employed) to enable one to arrive at the common principle by induction. The loadings of the task already measured should remain invariant.

In the interpretation of the factors the reader must keep in mind

Table 5

Matrix F_f = Matrix $F_c \cdot$ Matrix λ
Loadings on rotated factors

Variable	A	B	C	D	E	F	G	H	J	K
1	.084	.450	.027	.146	.155	–.174	–.381	.025	–.044	.211
2	.115	–.001	–.017	.558	.000	.024	.070	.517	.018	.222
3	.002	.081	.083	.555	.033	.045	–.014	.586	.104	–.040
4	.002	.003	.003	.632	–.002	–.002	.002	.588	.000	.004
5	–.099	–.017	.027	.416	.259	–.005	.004	.639	.002	.215
6	.191	.109	.688	.134	.123	.011	.072	–.034	.063	.202
7	.011	–.014	.725	.413	.083	.005	–.006	.020	.037	–.074
8	.028	.001	.810	.325	.026	.025	.027	.027	–.031	.133
9	–.037	–.066	.669	.170	.437	–.140	.016	–.017	–.086	.123
10	.449	.215	–.042	.228	–.025	–.032	.019	.481	–.019	–.109
11	.462	.216	–.104	–.157	.171	–.117	.129	.597	.006	–.112
12	.418	.088	.095	.085	.072	.005	–.055	.695	–.075	.031
13	.082	.005	–.006	.072	.509	.054	–.055	.527	–.126	–.021
14	.239	.011	–.040	.017	.320	.018	–.075	.726	–.061	.046
15	.692	–.007	.001	–.004	.003	–.002	.157	.005	.169	–.070
16	.216	.055	.030	–.001	.010	.004	.005	.102	.739	.013
17	.334	–.004	.003	.001	.010	.034	–.003	–.007	.754	.007
18	.018	–.012	.070	.029	.754	–.173	.013	.001	.033	.005
19	.455	–.002	.028	.056	.172	–.003	.026	.171	.297	.071
20	–.019	–.001	–.006	–.015	.012	.014	.630	–.001	.075	.080
21	.079	.047	.160	.025	–.157	–.024	.630	–.064	–.181	–.012
22	.202	–.008	–.001	.024	.569	–.031	–.005	.015	.027	.505
23	.109	–.006	–.034	.129	.455	.081	.308	–.062	.002	.520
24	.000	.091	.004	.053	.354	.028	–.022	–.093	.432	–.191
25	.072	.850	.015	.103	–.008	.002	.201	.030	–.097	–.007
26	–.026	.572	–.014	–.020	–.098	.264	.015	.146	.073	.440
27	.057	.769	–.010	–.149	.005	–.074	.275	.015	.081	–.012
28	–.034	.788	.140	.063	.059	.071	.019	–.014	.008	.230
29	–.048	–.027	.235	–.122	.220	.655	–.036	–.016	.012	.258
30	.048	.082	.005	.191	–.042	.744	.018	.063	.005	.010
31	.154	–.068	–.005	.103	.187	.636	.188	–.002	–.013	–.161
32	.332	.158	–.107	.204	.290	.036	–.267	–.010	–.035	.086

the direction of the measurements of errors, time consumption, etc., which is given on page 182.

Factor A. Inspection of column A in Table 5, reveals that twenty-two variables have projections of between $\pm .20$. Such loadings, contributing only .04 or less to the communality, may be considered as zero entries and will be here-in-after referred to as vanishing projections. Such a vanishing projection has no bearing on the interpreta-

tion of the factor in question. Five variables have loadings in excess of .400. They are:

(10)	"*A* errors" on elevated Maze *C*	+.449
(11)	"*B* errors" on elevated Maze *C*	+.462
(12)	"*C* errors" on elevated Maze *C*	+.418
(15)	"*A* errors" on elevated Maze *D*	+.692
(19)	Time to run elevated Maze *D*	+.455

For the purpose of interpretation these loadings must be considered of major importance. Some minor consideration should also be attributed to measures having projections of between .200 and .400. They are:

(14)	Time to run elevated Maze *C*	+.239
(16)	"*B* errors" on elevated Maze *D*	+.216
(17)	"*C* errors" on elevated Maze *D*	+.334
(32)	Feeding on Preliminaries	+.332

Inspection of the loadings shown above indicates immediately that, except for [32], they are all measures of good performance on the elevated mazes. In other words, animals which possess this ability in more than average degree will make fewer than average errors on these mazes and take less than average time. What abilities or factors will effect such improvements? Honzik (**5**, p. 11) states: "Vision, if it is present, assumes the dominant, almost the exclusive, role in the learning of an elevated maze of the type used." (His maze was not greatly different from Thorndike's elevated Maze *D*.)

A visual factor is therefore indicated; a factor which enables the animal to perceive visual patterns, react to visual cues; a visual-directional ability. This conclusion is further corroborated by the findings of Tsang. (**28, 29**).

It is a little more difficult to interpret the loading of variable [32] on this factor. One explanation might be that animals which rely predominantly on vision for orientation in the sensory milieu will tend to respond to distracting stimuli visually (that is, merely by looking toward the stimulating object) while continuing to feed; on the other hand, animals which are not so constituted and hence do not rely so much on vision will, when disturbed, leave the food and investigate at closer range by olfaction, vibrissae stimulation, or other taction.

Factor B. Twenty-five variables show vanishing projections on this factor. To be considered significant in explanation of the factor are the following:

(1)	Turns of the activity cage	+.450
(25)	Number of jumps in response to the buzzer signal in "C.R." apparatus	+.850

(26)	Number of jumps between buzzer signals	+.572
(27)	Number of jumps in response to the light signal in "C.R." apparatus	+.769
(28)	Number of jumps between light signals	+.788

Smaller loadings (probably insignificant) are:

(10)	"*A* errors" on elevated Maze *C*	+.215
(11)	"*B* errors" on elevated Maze *C*	+.216

The loadings indicated above are primarily those of the "conditioned response" apparatus. A reading of the report of Warner's original experiment and also of Thorndike's repetition of it leads one to believe that very little "conditioning" takes place in many animals. Those experienced in handling rats know of their "explosive" and purposeless behavior when shocked. This common observation, together with the fact that loadings on [26] and [28] are, for all practical purposes, as large as those on [25] and [27], leads one to doubt that any appreciable association of the signal with the shock is established. This factor, which is indicated by the above loadings, appears, then, to be a measure of wildness or panicky behavior, a measure of activities which are the results of fear.

Interestingly enough, Vaughn (**30**) identified a wildness-timidity trait in his study, even though his factor does not appear to be so clear-cut.

This fear may also cause the animal to avoid "*A*" and "*B*" errors on the first elevated maze by intensifying its attention to the available cues.

This explanation fits in nicely with interpretation of [1] included in this factor. Dr. Lashley told this writer that wild trapped animals, when put into an activity cage, must be watched carefully lest they "run themselves to death."

Factor C. Twenty-seven of the variables have vanishing projections on this factor. The significant ones are:

(6)	"*A* errors" in alley Maze *B*	+.688
(7)	"*B* errors" in alley Maze *B*	+.725
(8)	"*C* errors" in alley Maze *B*	+.810
(9)	Time to run alley Maze *B*	+.669

A small loading which might or might not be significant is

(29)	Number of approaches to the grid in the Columbia Obstruction Apparatus	+.235

This writer is at a loss to attempt any psychological interpretation of this factor. Honzik's (5) study shows that when an animal is deprived of vision and is faced with a problem normally solved by the aid of vision, it first falls back on olfaction for sensory cues. Factor *C* might represent an olfactory factor. Thorndike's report does not indicate that there were any controls to eliminate it.

On the other hand, the factor might have to be interpreted in terms of an ability displayed by the animal when faced with a double alternating alley maze. It should be remembered that Hunter (8) found that the animals had great difficulty solving this principle as such.

Factor D. This one is the *bête noire* of the factors. As it stands, twenty-four of the variables have vanishing projections on it. The variables which unquestionably have high loadings on it are:

(2)	"*A* errors" in alley Maze *A*	+.588
(3)	"*B* errors" in alley Maze *A*	+.555
(4)	"*C* errors" in alley Maze *A*	+.632
(5)	Time to run alley Maze *A*	+.416
(7)	"*B* errors" in alley Maze *B*	+.413

Some other and smaller loadings which may help in interpreting are:

(6)	"*A* errors" in alley Maze *B*	+.134
(8)	"*C* errors" in alley Maze *B*	+.325
(9)	Time to run alley Maze *B*	+.170
(10)	"*A* errors" on elevated Maze *C*	+.228
(32)	Feeding on Preliminaries	+.204

As the configuration is thus described the hyperplane defining zero loadings on Factor *D* has a number of large negative cosines with the other reference axes. These negative cosines mean that the hyperplane described is turned up into the positive manifold. If the cosines were reduced by turning this hyperplane away from *J* and *A*, the variables with significant loadings on these factors would also take a considerable loading on *D*. In turn, this would mean that *all* the error scores on all four mazes would take on some loading. The objectionable feature is that it would leave a configuration which was rather ill-defined. After inspection of both alternatives, the first-mentioned one was selected. This choice allows the variables having loadings on Factor *C* to take on loadings on *D* also, but excludes the ones on *A* and *J*.

Inspection of the variables having significant loadings on this factor discloses that they are all temporally located at the beginning of the experimental routine and that these loadings decrease progressively as time goes on and as the animals accustom themselves to their

environment. The animals which possess this ability adapt relatively quickly to new situations and, by so doing, are able to bring other abilities into play, thereby making comparatively few errors at the beginning of the experimental period. Such a factor is probably of an emotional rather than a cognitive nature and would be closely allied to the general progressive adaptation so prominently mentioned by Jackson (**10**) and by Alm (**1**). It has a counterpart in humans which is implied when we speak of children as having become "test wise." If the scores had been made by human subjects, this factor might have been termed "self-confidence."

The manifestation of this factor is probably the one which caused Tolman and Nyswander (**24**) to postulate that retracing was due to nervousness and caution, and the fact that [7] and [8] have significant loadings, whereas [6] and [9] do not, fits in with this line of reasoning.

Factor E. Twenty-two variables have very small or no loadings on this factor. Those to be considered are:

(5)	Time to run alley Maze *A*	+.259
(9)	Time to run alley Maze *B*	+.437
(13)	Time to start running elevated Maze *C*	+.509
(14)	Time to run elevated Maze *C*	+.320
(18)	Time to start running elevated Maze *D*	+.754
(22)	Time to start running in Jenkins Circular Problem Box	+.569
(23)	Time to run Jenkins Circular Problem Box	+.455
(24)	Time to reach food compartment in Latch Box Problem	+.354
(32)	Feeding on Preliminaries	+.290

All the above measurements are of time consumed to do some task. Animals possessing this factor in a more than average degree will take less than average time to perform the above tasks.

Some smaller loadings on this factor are:

(19)	Time to run elevated Maze *D*	+.172
(29)	Number of approaches to grid in Columbia Obstruction Apparatus	+.220

The ability indicated may be called a "speed factor" as Vaughn (**30**) termed it, but this name does not seem to do justice to the concept even though its end product is not incommensurable with speedy action. Possibly underlying the behavior measured here is a differential of motility of an innate character. Leeper (**14**) and

Rundquist (**18**) have shown that there is a hereditary aspect about this difference in rate of auto-kinesis.

This factor has been anticipated or surmised by a number of investigators. Time measurements have been subject to published speculation by, among others, Hicks (**4**), Hubbert (**7**), Davis and Tolman (**3**), Tolman and Nyswander (**24**), for animal learning, as well as Husband (**9**), Snoddy (**19**), Tinker (**23**) and Tryon (**26**), for human learning.

It should be noted that with the exception of variable [18], each one of these measures shows significant loadings on at least one other factor; that is, time scores reflect speediness of the animal together with the time consumed in making various kinds of errors. Both of these have been partialed out in the present study. Time scores were analyzed by partial correlation technique by Tolman and Nyswander (**24**). Their conclusion, based on an incomplete realization of the principles involved, was that time scores were ambiguous and should not be used. The present study indicates that, while raw time scores are ambiguous, they are no more so than raw error scores if these also have loadings on several factors.

Factor F. Twenty-eight variables have insignificant loadings on this factor. The significant ones are all connected with the Columbia Obstruction Apparatus.

(29)	Number of approaches to the grid	+.655
(30)	Number of contacts with the grid (resulting in shock)	+.744
(31)	Number of complete crossings of grid	+.636

A smaller loading is found on

(26)	Number of crossings between the buzzer signals in the Conditioned Response Apparatus	+.264

It is difficult to make psychological interpretations of such loadings, but, so far as this battery is concerned, the factor seems to be specific to the Columbia Obstruction Apparatus. The apparatus was designed to measure strength of drive under punishment, and this interpretation is possibly as close as the factor can be defined at present.

Factor G. Twenty-five variables have none of their variance to be accounted for by this factor. Those which have large loadings are:

(20)	Number of quadrants entered in Jenkins Circular Problem Box	+.630
(21)	Perfect trials in the Jenkins Circular Problem Box	+.630
(1)	Number of turns on Revolving Wheel Activity Cage	—.381

Other smaller loadings are:

(23)	Time to run Jenkins Circular Problem Box	+.308
(25)	Crossings to the buzzer in Conditioned Response Apparatus	+.201
(27)	Crossings to the light in above apparatus	+.275
(32)	Feeding on Preliminaries	—.267

In this case, also, psychological interpretation must be made with care. Strictly speaking, all that can be said about this factor is that it is specific to the Jenkins Circular Problem Box; yet the writer deems it important that this factor also splits the scores on the Conditioned Response Apparatus in such a way that the scores which indicate understanding of problems [25] and [27] have loadings, whereas the other scores on this same piece of apparatus (but indicating only erratic and random behavior under the stimulation of fear) have no loadings at all. This factor appears to indicate the extent to which specific association between signal and response was established.

This factor is also the only one in which a significant negative loading (on variable [1]), is observed.

What, then, shall we say of animals which: (1) enter few quadrants, have many perfect trials and take short time in the Jenkins Box; (2) make relatively few turns on the activity cage; (3) within the limited training time on the Conditioned Response Apparatus, learn to cross at the right time? They seem to display some kind of a cognitive, possibly a logical, factor, a generalized problem-solving ability and perhaps an alertness to sensory cues. The presence of [32] in negative form cannot be explained by this writer; it might be fortuitous.

Factor H. Twenty-three variables have vanishing loadings on this factor. Nine variables have significant loadings. They are:

(2)	"*A* errors" in alley Maze *A*	+.517
(3)	"*B* errors" in alley Maze *A*	+.586
(4)	"*C* errors" in Alley Maze *A*	+.588
(5)	Time to run Alley Maze *A*	+.639
(10)	"*A* errors" on elevated Maze *C*	+.481
(11)	"*B* errors" on elevated Maze *C*	+.597
(12)	"*C* errors" on elevated Maze *C*	+.695
(13)	Time to start on elevated Maze *C*	+.527
(14)	Time to run on elevated Maze *C*	+.726

Obviously, the common element in this case is the principle of turning alternately right and left. The discovery of this working principle seems well within the grasp of the animals. An "alternating hypothesis" was described by Krechevsky **(11)**. Miles **(17)** has

shown that the principles or factors used in the alley maze are also made use of in the elevated maze and vice versa.

Factor J. Twenty-eight variables have vanishing projections on this factor. Only three variables have loadings above .400, and one has a loading of about .300. The variables with significant loadings are:

(16)	"*B* errors" on elevated Maze *D*	+.739
(17)	"*C* errors" on elevated Maze *D*	+.754
(24)	Time to reach food in Latch Box	+.432

Another fairly high loading is

(19)	Time to run elevated Maze *D*	+.297

It is difficult to differentiate this factor from *A*. Both appear to involve vision. Yet *J* is independent of *A*. Dr. Thorndike told the writer that "*B*" and "*C*" errors on the *D* Maze were relatively rare. The most plausible interpretation seems to point to "relationships visually perceived," perhaps a "visual insight" factor, as distinct from *A*, which seems to operate as a "visually directing" factor at choice points. Honzik's data (**5**) on the function of vision in maze learning, and Honzik and Tolman's data (**6**) on the role of vision in spatial relationship, both indicate that more than one factor enters into the observed behavior. One of these, *A*, may be tentatively thought of as an ability to benefit from sensory cues at the various choice points, while the other, *J*, may be thought of as a visual perception of the total maze gestalt.

Factor K. The hyperplane defining absence and presence of significant loadings on this factor is not very well determined. It may be that this factor is contaminated by factors not extracted. There are twenty-three variables having negligible loadings. On the other hand, only three variables have loadings in excess of .400. They are:

(22)	Time to start running in the Jenkins Circular Problem Box	+.505
(23)	Time to run above-mentioned apparatus	+.520
(26)	Number of crossings between buzzer signals in Conditioned Response Apparatus	+.440

This leaves six variables having loadings between .200 and .300. They are:

(1)	Number of turns in activity cage	+.211
(2)	"*A* errors" in alley Maze *A*	+.222
(5)	Time to run alley Maze *A*	+.215
(6)	"*A* errors" in alley Maze *B*	+.202

(28)	Number of crossings between light signals in Conditioned Response Apparatus	+.230
(29)	Number of approaches to the grid in Columbia Obstruction Apparatus	+.258

This is probably a residual factor as it stands, and no attempt will be made to interpret it psychologically even though there seems to be some consistency in the loadings.

Factors and Apparatus. A survey of the battery in terms of the concomitance of primary abilities and the apparatus employed as measurement will be attempted. It would seem pertinent to think of this when selecting apparatus for further research.

The activity wheel cage, as has been stated, is a reliable measuring instrument, but, except for loadings on *B* (tendency toward panic) and *G* (cognitive?) factors, the behavior exhibited in this apparatus is not sufficiently like that measured by other tests in the battery to be distinctly analyzed. Much of the variance remains to be accounted for.

The Warner-Warden single alternating maze involves two factors previously described, the *D* (general adaptation or confidence) and the *H* (single alternation principle) factors. The time measurement reflects both these factors and, in addition, the general motility factor, *E*.

The Warner-Warden double alternating maze yields scores which define a Factor *C*. This appears to be specific to this apparatus in so far as this battery is concerned. The time measurements behave as indicated above.

The elevated mazes seem to be primarily solved by the visually-directing ability, Factor *A*. If the maze also involves a single alternating principle, Factor *H* also may be involved. Apparently a high degree of familiarity with elevated mazes leads to a shift from behavior depending upon visual orientation at each choice point to an insight in terms of the total visual gestalt (Factor *J*).

The Jenkins Circular Problem Box has a factor specific to it in so far as this battery is concerned. A guess might be made that this is some cognitive factor, but such a guess is based on inadequate loadings.

The Latch Box presents a dilemma similar to that of the Activity Cage. The large difference between the communality and the reliability indicates that the ability to solve this problem is not utilized in other tasks represented in the battery. There is a loading on the visually-perceptive Factor *J*, but it accounts for only a small portion of the total variance of this test.

It turns out that the so-called Conditioned Response Apparatus measures principally a tendency for hyperactivity under fear. This is probably the same factor generally referred to as wildness. There are some slight loadings of the relevant responses on a factor tentatively identified as cognition, *G*.

The Columbia Obstruction Apparatus employs a factor which in the battery under consideration is specific to this piece of apparatus.

The writer has lithoprinted copies of the 45 plots (each factor with every other one), together with a plot of the tenth factor residuals. They may be obtained on request.

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FACTOR ANALYSIS AND THE INDEX OF CLUSTERING*

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The initial problem of factor analysis is described as a search for clustering of the test vectors. Curves are developed which give a visual picture of the clustering tendency, and an index of clustering is derived which provides a simple estimate for the number of factors.

It is well known that the same test given to the same person a number of times leads to a set of scores which may have considerable dispersion. By definition the true score of a person in a test is the average of an infinite number of repetitions. In practice this has to be amended for obvious reasons, but the principle is clear: any test score is regarded as drawn from a population of scores of which the mean is the true score. A sample score is the sum of the true score and an error score from an error population with zero mean.

I

If a factor analysis is to be run on a group of tests and a group of individuals, the ideal procedure would be to repeat the testing an infinite number of times and average the scores to obtain the true scores. Suppose to begin with that this procedure has been followed, and let S_{ij} be the true score in test i $(i = 1 \cdots n)$ of individual j $(j = 1 \cdots N)$. Suppose $N > n$. The n tests represent n vectors in the population space of N dimensions. The vectors themselves lie in a subspace η of not more than n dimensions, and factor analysis is concerned with the configuration of the vectors in this subspace. They might stick out in all directions like pins around a cushion ball, or they might tend to cluster together so as to be approximately contained in a still smaller subspace R of dimensionality r which is less than that of η.

For example consider a case in which η is 2-dimensional. The vectors might be clustered around a preferred direction as in Fig. 1, so that such an R of lower rank can be found; or they might be spread

* Based, in part, on a paper "Matrix Approximation Criteria" read at the District meeting of the Psychometric Society, Chicago, April 3, 1937. The author wishes to acknowledge suggestions gained in conversations with Dr. A. S. Householder and Mr. Clyde Coombs, of the University of Chicago.

around as in Fig. 2, in which case there is no such R. In the first case clustering is present; in the second it is absent.

FIGURE 1

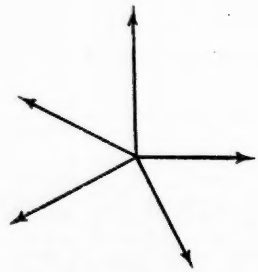

FIGURE 2

In 2- or 3-dimensional spaces clustering can be detected by simply plotting the vectors and looking for it; in higher-dimensional spaces, this method is not possible, and the degree of clustering must be ascertained by analytical methods. The straightforward method of procedure is to see how nearly the vectors are contained in the best fitting one-dimensional subspace, then how nearly they are contained in the best 2-space, etc. At any stage the size of the residuals, representing the differences between the original vectors and their projections into the best subspace, shows how nearly the test vectors are clustered into a space of that dimensionality.

Let $\phi(d)$ be the sum of the squares of the residuals for the best subspace of d dimensions. Then the graph of ϕ against d gives a visual picture of the clustering tendency. At $d = 0$, it is equal to the sum of the squares of the scores S_{ij}, and it drops monotonically to zero for some value of d equal to or less than n. Further, the curve is always concave upward, so that it descends more steeply at first than it does later.

The clustering can be seen by merely looking at this curve. For example, in Fig. 3 the vectors are practically all congregated together in an r-dimensional space, and one would stop after taking out r factors in the analysis; in Fig. 4 there is no such stopping point. This can perhaps be seen even more clearly by considering $\lambda(d) = \phi(d) - \phi(d-1)$, which quantity measures the importance of taking out the d-th factor. Thus in Fig. 3a all the factors beyond the r-th are negligible compared with the first r. In Fig. 4a, however, they are all the same size and there is no justification for retaining some and dropping others.

Now $\lambda(d)$ is the d-th largest latent root of $A = SS'$, so that the configuration represented in Fig. 4 and Fig. 4a is a completely spheri-

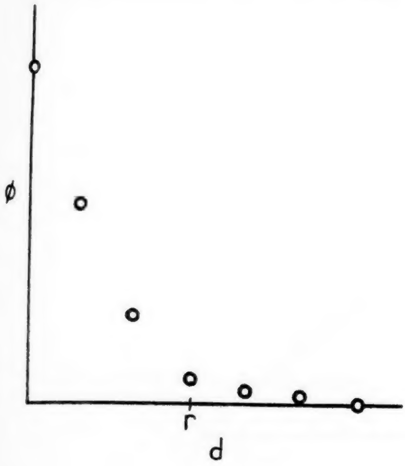

FIGURE 3

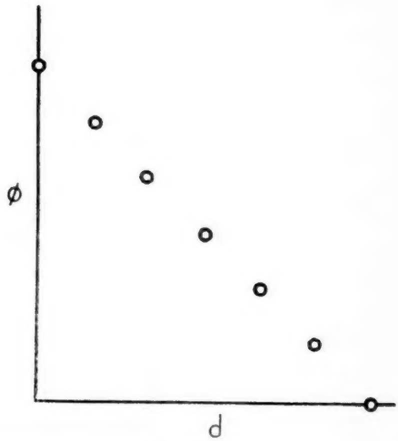

FIGURE 4

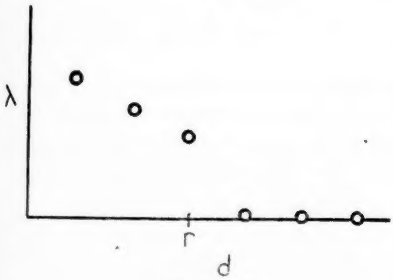

FIGURE 3*a*

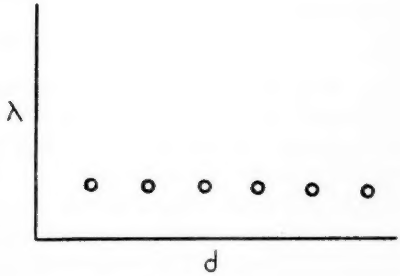

FIGURE 4*a*

cal one, in that its principal axes are all of equal length. There is thus no preferred direction at all, and the vector arrangement is completely random. The other extreme from complete randomness is complete clustering, in which the vectors would all lie along the same line. Fig. 1 and Fig. 2 approximately represent these extreme cases. In Fig. 5 the upper line represents a completely random configuration, the lower line a completely clustered one. It is seen that the greater the clustering the smaller the area under the ϕ curve.

The picture of the clustering tendency afforded by the curves above is clear and complete, and is available in actual factoring to show when to stop taking out factors. Thus the factoring may be stopped when the next λ (these quantities decrease monotonically with d) has become negligible compared with the preceding ones.

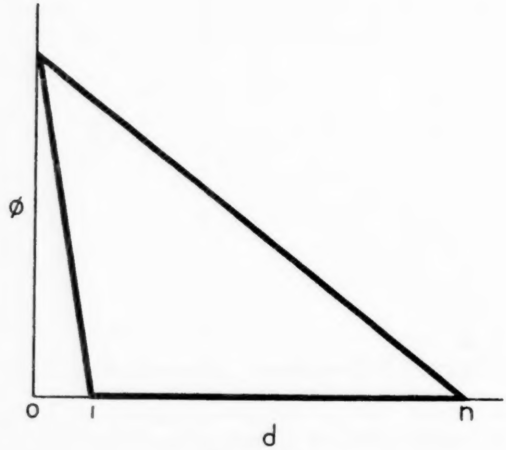

FIGURE 5

II

It is convenient to have some measure of the clustering which can be readily applied to any given matrix S without going to the labor of calculating the above. What is wanted is some measure of the inequality of the latent roots of $A = SS'$. For this purpose we may consider their dispersion, namely

$$\alpha = \sum_{i=1}^{n} (\lambda_i - \bar{\lambda})^2 , \tag{1}$$

where

$$n\bar{\lambda} = l = \sum_{1}^{n} \lambda_i = \sum_{1}^{n} a_{ii} = \sum_i \sum_j s^2{}_{ij} . \tag{2}$$

This quantity vanishes when S represents a completely random vector arrangement, and increases to

$$b = (1 - \frac{1}{n}) l^2 \tag{3}$$

as the vectors swing into a completely clustered configuration. Hence

$$\zeta = \frac{\alpha}{b} \tag{4}$$

is a quantity which increases from 0 to 1 as the matrix varies from complete randomness to complete clustering. It can be shown that ζ is also given by

$$\zeta = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_i - \lambda_j)^2}{2(n-1)l^2}. \tag{5}$$

The set of latent roots λ_i, or rather their relative sizes λ_i/l, completely specifies the clustering. The same is not strictly true of ζ; for example, the following two sets of numbers both have $l = 6$ and $\zeta = 0.0825$:

Set no. 1	3	2	1
Set no. 2*	3.15	1.5	1.35.

The reason for this may be seen by expanding (1) to obtain

$$\alpha = \sum \lambda_i^2 - \frac{1}{n} l^2 \tag{6}$$

from which it is clear that ζ involves only the first two power moments of the λ_i, whereas the first n moments would be required to completely specify the λ_i. Note that

$$\zeta = \frac{n}{n-1} \frac{\alpha}{l^2} \tag{7}$$

which, apart from a factor involving n, is simply the relative dispersion of the λ_i.

However ζ does roughly increase monotonically with the degree of clustering, and hence may be called an *index of clustering*. To see this it is necessary to consider the behaviour of α in greater detail. Given a set of λ_i and consequently a value of α, let two of them, say λ_k and λ_m, change equally and oppositely while the others remain constant. This keeps their sum l also constant. Under these conditions the change in α with respect to λ_k is found by differentiating (6) to be

$$\frac{\partial \alpha}{\partial \lambda_k} = 2(\lambda_k - \lambda_m). \tag{8}$$

Now start with a completely random matrix so that the λ are all equal. We may reach any other set of λ with the same sum and arranged in decreasing (from the left) order of magnitude by decreasing λ_n to its final value and correspondingly increasing λ's at the left end, then decreasing λ_{n-1}, etc. Let λ_k be the decreasing root and λ_m the increasing one. Since always $\lambda_m \geqslant \lambda_k$ during such a process, we see by (8) that

*More exactly,

$$\frac{4.5 + \sqrt{3.25}}{2}, 1.5, \frac{4.5 - \sqrt{3.25}}{2}.$$

α continually increases. The more the roots at the left gain, the higher the clustering (see Figs. 3a and 4a) and the larger is α. Thus ζ is monotonic with the clustering.

For the simple type of configuration in which $n - p$ of the roots are zero, while the other p are equal(which would be clearly a p-factor case), ζ has the value

$$\zeta(p) = \frac{n-p}{p(n-1)} \tag{9}$$

from which

$$p = \frac{n}{1 + (n-1)\zeta}. \tag{10}$$

If n is considerably larger than p, (9) is approximately

$$\zeta(p) \simeq \frac{1}{p}, \tag{11}$$

and in this case the number of factors is simply $1/\zeta$.

If the matrix is known to have t zero roots, then its vectors are arranged randomly in the remaining $(n - t)$-space if

$$\zeta = \frac{t}{(n-1)(n-t)}. \tag{12}$$

Any excess over this value is a sign of clustering. This may be used to test a residual matrix for randomness after t factors have been removed from the original score matrix. The ratio of the excess to [unity — (12)] gives a measure of the clustering of the residuals in the residual space.

The quantity ζ can of course be employed to study the clustering of any sub-set of vectors of the configuration, and hence may be useful in picking out sub-clusters or constellations, etc.

Having indicated some uses of the ζ function,* we turn now to its calculation. Since α is a symmetric function of the roots of the characteristic equation of $A = SS'$, it can be expressed in simple manner in terms of the coefficients of that equation. In fact, if the equation is

$$\lambda^n - l\,\lambda^{n-1} + h\,\lambda^{n-2} + \cdots = 0, \tag{13}$$

then†

* Any power or root of ζ would vary from 0 to 1 monotonically with ζ, and hence would also serve as an index. In an application to theory of single tests it has been found that $\sqrt[4]{\zeta}$ makes close contact with the usual index of reliability (Young, Gale and Libby, J. E. P. On the index of reliability. Unpublished manuscript).

† Dickson, L. E. First course in the theory of equations. New York: John Wiley & Sons, 1922.

$$\sum \lambda_i^2 = l^2 - 2\,h\,. \tag{14}$$

Now l is simply the quantity defined in (2), while h is the sum of all principal 2×2 minors of A, namely

$$\begin{aligned} 2h &= \sum_i \sum_j \left(a_{ii}\,a_{jj} - a_{ij}^2\right) \\ &= l^2 - \sum \sum a_{ij}^2\,. \end{aligned} \tag{15}$$

Combining (6), (14), (15), and the definition of ζ we finally obtain

$$\zeta = \frac{n \sum \sum a_{ij}^2 - l^2}{(n-1)\,l^2}\,. \tag{16}$$

In other words, the calculation of ζ requires the sum of the diagonals of A, and the sum of squares of all its elements. This last quantity may perhaps be approximated by calculating the mean $\bar{a}$ of the a_{ij}, estimating in some manner their dispersion, σ^2, and using $(\bar{a})^2 + \sigma^2 = \overline{a^2}$. Or this might be done only for the off-diagonal elements, the sum of squares of the diagonals being computed exactly.

III

We may illustrate the above considerations by applying them to a matrix recently studied by Mosier.* He took an apparently 4-factor 20×4 size matrix V_0, formed $R_0 + V_0\,V_0'$, sprinkled error components into R_0 to get another matrix R_1, and then factored R_1 to see how nearly he got back to the configuration of V_0. This was done to afford a comparison of various criteria for the factoring process.

Now in such a study as this it is important to know just what the configuration of the starting matrix V_0 is; otherwise it is hardly possible to judge criteria for getting back to it. Since the non-zero latent roots of $V_0\,V_0'$ are the same as those of $V_0'\,V_0$, it suffices to consider the latter smaller matrix. Its ζ value is readily found to be 0.104, from which (10) gives, with $n = 4$, $p = 3.05$; i.e., it is indicated that V_0 represents only a 3-factor matrix. To check this finding, the latent roots were computed and found to be 6.15, 5.22, 2.40, and 1.08. Thus 2 factors will account for 76% of the variance of V_0, while 3 factors will account for better than 92%. In other words, the 4-th factor represents but 7% of the total variance, confirming the above indication that V_0 is essentially only a 3-factor matrix.

Comparing with row 6 of Mosier's Table III, which represents

* Mosier, C. I. Influence of chance errors on simple structure, *Psychometrika*, 1939, **4**, 33-44.

a rough set of λ values, we find that the first 4 values, 6.28, 5.12, 2.03, and 1.57, agree roughly with the latent roots as given above; and hence that his criterion No. 6 does give a fair indication of what was really in his starting matrix V.

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A FACTOR ANALYSIS OF THE ORIGINAL STANFORD-BINET SCALE*

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From the original Stanford-Binet scale, those items passed by between 10 and 90 per cent of a group of ten-year-old children were analyzed by the centroid method. Upon rotation, there appeared a common factor, for which two explanatory hypotheses are offered, the more tenable being that it is an effect of maturation. Primary factors tentatively identified are Number, Space, Imagery, Verbal Relations and Induction. A sixth factor apparently involves a reasoning ability and a seventh can not be interpreted.

This study is an attempt to isolate the primary abilities in a part of the original Stanford-Binet Scale (**3**), to identify them if possible, and to determine the factorial content of each test item under consideration.

It was desired to have a population of subjects whose mental development as measured by mental age varied widely. In order to keep the subjects as homogeneous as possible, chronological age was held constant between the limits of ten years and ten years, eleven months, inclusive. This age was selected as one which should give a wide range of performance in the middle part of the scale and as one for which individual differences, if any, in the rate of mental development should be noticeable.

The data were obtained from the Institute for Juvenile Research, Chicago, Illinois. The records of all ten-year-old children who were tested at the Institute during the period from 1927 to 1937, inclusive, were analyzed to determine which items should be used as variables. It was impossible to use the entire scale since the actual test performance of a population of individuals of a given age would not include all the items. The criterion for the selection of items was based on the proportion of successes and failures. Only those items which were passed by more than 10 per cent and by less than 90 per cent of the population were included. After this process of selection, thirty-one items were retained. The addition of mental age brought the number of variables to thirty-two.

* The writer is indebted to Dr. L. L. Thurstone for his interest and assistance throughout this study and to Dr. Andrew W. Brown, who made possible the collection of data at the Institute for Juvenile Research, Chicago, Illinois.

TABLE 1

Correlational Matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	.68														
3	.48	.49													
4	.79	.67	.42												
5	.55	.47	.41	.49											
6	.60	.47	.43	.52	.52										
7	.71	.66	.50	.64	.59	.63									
8	.72	.65	.63	.60	.54	.67	.72								
9	.74	.63	.45	.80	.50	.51	.61	.66							
10	.53	.54	.41	.63	.36	.55	.60	.56	.46						
11	.67	.73	.47	.81	.45	.58	.67	.61	.71	.71					
12	.69	.57	.55	.80	.55	.56	.59	.63	.67	.62	.66				
13	.80	.69	.56	.77	.64	.69	.71	.79	.73	.52	.72	.75			
14	.61	.54	.41	.67	.51	.59	.62	.71	.63	.48	.51	.59	.75		
15	.76	.65	.58	.64	.62	.70	.69	.85	.63	.58	.61	.71	.75	.67	
16	.71	.72	.60	.70	.68	.77	.74	.78	.58	.61	.65	.65	.80	.67	.79
17	.51	.75	.41	.56	.32	.28	.56	.50	.41	.45	.40	.50	.56	.32	.35
18	.82	.60	.55	.82	.46	.57	.76	.81	.76	.58	.76	.67	.73	.71	.78
19	.69	.56	.44	.49	.60	.61	.56	.61	.64	.50	.55	.60	.73	.52	.74
20	.59	.58	.47	.68	.52	.60	.71	.64	.52	.48	.59	.53	.62	.52	.60
21	.50	.63	.50	.59	.65	.44	.70	.85	.48	.50	.44	.57	.57	.65	.85
22	.63	.63	.47	.45	.55	.52	.65	.75	.55	.45	.48	.65	.69	.55	.78
23	.54	.70	.85	.50	.65	.45	.50	.63	.47	.54	.61	.59	.56	.48	.59
24	.65	.60	.47	.75	.58	.55	.70	.75	.58	.48	.56	.62	.72	.63	.76
25	.73	.55	.46	.72	.67	.67	.57	.78	.63	.55	.62	.70	.71	.63	.76
26	.68	.55	.40	.80	.51	.57	.64	.69	.58	.54	.67	.86	.70	.60	.61
27	.55	.57	.61	.60	.67	.65	.60	.70	.53	.54	.62	.64	.72	.53	.63
28	.75	.60	.50	.67	.60	.68	.70	.75	.61	.54	.64	.64	.73	.70	.73
29	.60	.54	.47	.65	.60	.47	.75	.70	.53	.52	.50	.63	.76	.50	.61
30	.70	.58	.75	.55	.55	.38	.68	.80	.49	.60	.45	.45	.60	.46	.75
31	.70	.60	.75	.58	.60	.50	.65	.80	.65	.62	.47	.52	.68	.55	.53
32	.90	.88	.72	.84	.78	.76	.83	.91	.80	.68	.82	.80	.91	.81	.87

The variables are described below in the order of their code numbers:

1. (VII -3) Repeating five digits
2. (VII -6) Copying diamond
3. (VIII -1) Ball and field, inferior plan
4. (VIII -2) Counting backwards from 20 to 1
5. (VIII -3) Comprehension
6. (VIII -4) Finding likenesses between 2 items
7. (VIII -5) Definitions superior to use
8. (VIII -6) Vocabulary
9. (IX -1) Giving date
10. (IX -2) Arranging five weights
11. (IX -3) Making change

TABLE 1 (continued)
Correlational Matrix

	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
2																
3																
4																
5																
6																
7																
8																
9																
10																
11																
12																
13																
14																
15																
16																
17	.41															
18	.61	.44														
19	.68	.35	.57													
20	.53	.44	.64	.49												
21	.76	.36	.65	.78	.52											
22	.70	.24	.74	.73	.61	.73										
23	.69	.38	.46	.59	.46	.38	.38									
24	.60	.45	.80	.73	.54	.85	.71	.40								
25	.77	.49	.73	.66	.58	.78	.76	.46	.71							
26	.68	.39	.70	.64	.56	.46	.55	.36	.63	.53						
27	.68	.33	.60	.57	.64	.52	.47	.47	.54	.58	.56					
28	.80	.39	.62	.67	.60	.73	.54	.43	.67	.62	.59	.50				
29	.58	.41	.57	.48	.50	.60	.61	.52	.65	.70	.61	.32	.57			
30	.68	.46	.61	.63	.39	.53	.61	.40	.41	.52	.42	.51	.52	.46		
31	.60	.50	.58	.53	.59	.48	.52	.28.	63	.58	.57	.46	.48	.63	.30	
32	.85	.58	.86	.71	.79	.85	.86	.75	.90	.88	.80	.83	.80	.83	.86	.84

12. (IX -4) Repeating four digits backwards
13. (IX -5) Three words in one sentence
14. (IX -6) Rhymes
15. (X -1) Vocabulary
16. (X -2) Verbal absurdities
17. (X -3) Drawing designs from memory
18. (X -4) Reading and report
19. (X -5) Comprehension
20. (X -6) Naming sixty words in one minute
21. (XII -1) Vocabulary
22. (XII -2) Definitions, abstract words
23. (XII -3) Ball and field, superior plan
24. (XII -4) Dissected sentences
25. (XII -5) Interpretation of Fables

26. (XII -6) Repeating five digits backwards
27. (XII -7) Interpretation of pictures
28. (XII -8) Finding likenesses, three items
29. (XIV -2) Induction test, paper cutting
30. (XIV -4) Problem questions
31. (XIV -6) Reversing hands of clock
32. Mental age

Records of subjects who passed nearly all or who failed nearly all of these items had no value for the analysis and were discarded. Four hundred fifty-six subjects had a sufficient number of items within the specified range and were included in the study.

The intercorrelation coefficients of the thirty-two variables were computed. Tetrachoric coefficients were used. Computational labor was reduced by the use of facilitating tables (**1**). The resulting intercorrelations are shown in Table 1. The coefficients were all positive, as was to be expected from the nature of the material, and all were .24 or higher.

The correlational matrix was factored by the centroid method, following the procedure outlined in the appendix of *Vectors of Mind* (**6**, pp. 232-250) with one exception. After the sign change in the extraction of the second factor, reflection of the variables was effected by a more complete method (**6**), which increases the amount of total variance accounted for by each successive factor.

The number of factors which were extracted from the correlational matrix was determined by applying Tucker's empirical rule (**5**, pp. 66-67) that when all the significant factors have been extracted so that only chance variation remains in the residuals, an empirical criterion ϕ takes a limiting value. ϕ is defined by the equation

$$\phi = \sqrt{\frac{\sum P_{S+1}}{\sum P_S}},$$

where $\sum P_S$ is the sum of the absolute values of the residuals after S factors, and $\sum P_{S+1}$ is the sum of the absolute values of the residuals after $S+1$ factors. The summation includes the diagonal entries; $\sum P_S$ includes the adjusted diagonals and $\sum P_{S+1}$ includes the computed diagonals. The limiting value which ϕ approaches is

$$\frac{n-1}{n+1}$$

where n is the number of variables in the correlational matrix. The limiting value of ϕ for thirty-two variables is .939. After six factors were extracted, the value of ϕ approached the limiting value and after seven factors were extracted, the value was .952. Since .952

TABLE 2

Centroid Matrix

	I	II	III	IV	V	VI	VII
1	.854	—.065	—.127	.118	.104	.102	—.083
2	.794	—.246	.129	—.248	—.146	.242	.078
3	.679	—.182	.310	—.375	.320	—.189	—.207
4	.837	—.297	—.244	.204	—.179	.228	—.113
5	.715	.131	.153	—.081	—.080	—.252	—.051
6	.726	.079	.092	.299	.097	—.166	.290
7	.838	.052	—.060	—.160	—.040	.166	.185
8	.903	.164	—.038	—.149	.217	—.028	.111
9	.769	—.152	—.155	.148	.091	.078	—.158
10	.694	—.154	.122	.055	—.107	.072	.076
11	.778	—.307	.085	.245	—.099	.163	—.034
12	.812	—.234	—.103	.192	—.160	—.226	—.152
13	.898	—.052	—.106	.087	.063	—.143	.058
14	.753	.099	—.087	.144	.066	.041	.108
15	.877	.272	.108	.085	.075	.047	—.121
16	.872	.105	.256	.124	—.086	—.146	.171
17	.567	—.318	—.088	—.297	—.192	.166	.160
18	.852	—.021	—.263	.081	.168	.251	—.189
19	.769	.159	.146	.069	—.028	—.053	—.139
20	.729	—.083	—.056	—.063	.085	.029	.147
21	.786	.512	.066	—.110	—.284	.108	—.034
22	.772	.349	—.094	—.062	.083	—.078	—.147
23	.669	—.197	.473	—.138	—.123	—.176	—.243
24	.816	.226	—.207	—.049	—.147	.051	—.093
25	.834	.196	—.101	.073	—.060	—.117	—.152
26	.764	—.152	—.230	.199	—.089	—.129	.067
27	.739	—.103	.243	.112	.195	—.082	.102
28	.804	.153	.062	.174	—.082	.127	.139
29	.744	.077	—.245	—.230	—.223	—.196	—.057
30	.714	.147	.262	—.092	.202	.228	—.147
31	.740	—.150	—.211	—.273	.185	—.153	.171
32	1.044	.011	.032	—.164	.091	.066	—.071

slightly exceeded the limiting value, no more factors were drawn. The centroid matrix is shown in Table 2. The seventh factor residuals were tabulated and are presented in Table 3. They are consistently small and their range is between $\pm.169$.

The arbitrary orthogonal reference frame obtained by application of the centroid method to the correlational matrix was rotated into a unique position with reference to the co-ordinate axes in order that the axes should be scientifically meaningful. Each co-ordinate plane was rotated independently until a simple configuration of the test variables was overdetermined. The graphical method used in the rotation is essentially that described in *Primary Mental Abilities* (**5**,

TABLE 3

Distribution of Residuals

Residuals, Disregarding Signs	f
.000 to .009	83
.010 to .019	67
.020 to .029	83
.030 to .039	72
.040 to .049	50
.050 to .059	41
.060 to .069	24
.070 to .079	18
.080 to .089	18
.090 to .099	7
.100 to .109	12
.110 to .119	8
.120 to .129	5
.130 to .139	1
.140 to .149	3
.150 to .159	3
.160 to .169	1
Σ	496

pp. 23-28) except that the planes were not required to pass through the origin. The criterion for determining a simple configuration was the maximizing of the number of nearly vanishing entries in the factorial matrix. An entry was considered nearly vanishing if it was in the range $\pm .20$, since it accounts for 4 per cent or less of the variance and is within the range of chance errors. It was also desirable to avoid having any significant negative projections. Both of these stipulations were met by the entries for six of the factors, but plane *C*, represented by the third column of the rotated factorial matrix, Table 4, has no nearly vanishing entries. The lowest loading is .432 for variable 17. During the rotational procedure, this axis remained oblique until orthogonality was forced upon it by setting it orthogonal to the other six axes. The result was the appearance of the rotated factorial matrix, Table 4, and the complete orthogonality of the structure. The cosine matrix is shown in Table 5. The common factor was found to diverge about 23° from the first centroid. The matrix of the transformation from the centroid matrix to the rotated factorial matrix is shown in Table 6.

In the psychological interpretation of the factors, loadings within $\pm .20$ were disregarded as chance variations, and only loadings of .30 or above were considered in attempting to name a factor. A value of .30 is low for this purpose, but because of the complex nature of most of the variables and because of the large amount of variance

TABLE 4
Rotated Factorial Matrix

	A	B	C	E	G	H	J
1	.244	.022	.838	—.024	.108	.048	.088
2	.073	.344	.636	.070	.500	.232	.100
3	—.076	.370	.644	—.030	—.044	.605	—.088
4	.530	.086	.724	—.058	.304	—.046	.130
5	.070	.230	.608	.352	—.086	.276	.174
6	—.011	—.032	.678	.064	—.076	.046	.532
7	—.050	.217	.778	.166	.318	—.022	.123
8	—.136	.172	.924	.136	.043	.067	.091
9	.337	.026	.748	—.084	.068	.062	.055
10	.157	.140	.562	.063	.243	.182	.268
11	.381	.026	.636	—.080	.285	.202	.296
12	.506	.230	.666	.088	—.080	.156	.263
13	.174	.185	.844	.061	—.038	.056	.262
14	.072	—.016	.740	.085	.068	—.048	.219
15	.086	—.094	.856	.282	.046	.194	.084
16	.032	.106	.732	.292	.054	.252	.460
17	.072	.457	.432	—.004	.428	.017	.068
18	.272	—.054	.896	—.066	.164	—.050	—.109
19	.156	.008	.595	.286	.010	.255	.126
20	.004	.202	.692	—.009	.147	.035	.172
21	.004	.024	.696	.668	.207	.038	.019
22	.046	.034	.798	.337	—.100	.048	—.064
23	.208	.296	.463	.220	.078	.682	.144
24	.192	.115	.770	.358	.102	—.079	—.023
25	.243	.069	.786	.316	—.080	.064	.078
26	.333	.178	.678	.024	—.019	—.078	.307
27	.002	.059	.681	—.053	.027	.314	.342
28	.076	—.060	.727	.220	.210	.026	.301
29	.191	.429	.648	.371	—.016	—.026	—.001
30	—.064	—.086	.717	.116	.222	.341	—.072
31	—.072	.455	.738	—.064	.003	—.014	.083
32	.078	.222	.989	.153	.186	.231	.030

TABLE 5
Cosine Matrix

	A	B	C	E	G	H	J
A	.999						
B	—.065	.999					
C	.002	.000	1.000				
E	.006	.121	—.001	1.000			
G	—103	.006	.001	—.036	.999		
H	.057	.093	—.001	.049	—.030	.999	
J	—.042	—.002	.002	—.034	—.057	.009	1.000

TABLE 6
The Transformation Matrix

	A	*B*	*E*	*G*	*H*	*J*	*C*
I	.158	.174	.164	.130	.152	.186	.917
II	—.374	—.404	.670	—.206	—.236	—.170	.141
III	—.256	—.134	.152	.177	.831	.295	—.162
IV	.454	—.702	—.200	—.225	—.142	.526	.003
V	—.435	—.276	—.638	—.321	.089	—.192	.330
VI	—.063	—.433	—.202	.838	—.184	—.315	.062
VII	—.608	.176	—.116	.230	—.410	.661	—.024

which is accounted for by the common factor loadings, it was thought expedient to consider values of this magnitude.

The order of the columns in the rotated factorial matrix is of no significance. The third column will be considered first since it represents the most interesting development of the study: the appearance of a common factor. It has no nearly vanishing entries and no negative projections. All of the entries are significantly high, the lowest being .432 for variable 17. Two hypotheses will be considered in the discussion of these findings. The first hypothesis is that the factor is of the nature of Spearman's "*G*"; a general factor which "though varying from individual to individual remains the same for any one individual in respect to all the correlated abilities" **(2)**. If this interpretation is correct, the same general factor should appear in factor analyses of test data independently of the age of the subjects or of the particular test battery.

The second hypothesis is that the factor is maturation. This hypothesis is formulated on the bases that mental functions at first undifferentiated, develop from the general to the specific, and that there are individual differences in the rate of mental development. Either of these assumptions could account for the appearance of a common factor resulting from maturation. If, in young children, the primary abilities are not differentiated, a general approach involving several abilities will be used in solving a given problem. As differentiation progresses, the approach will tend to become more selective, until at the adult level, where complete independence of the abilities is postulated, the choice of approach will be restricted to the use of the most appropriate method. And, if in a population of children at the same chronological age, those children of a more accelerated rate of mental growth have all the abilities more advanced than children of a slower rate of growth, there would be a correlation existing between the primary abilities which would not be present in an adult population. The common factor may be the result of either of these effects or it may be a product of both. It is impossible to discriminate on the

basis of the present data. There is also the possibility that the common factor may be greatly influenced by the nature of the test battery, and by the original method of selection of the items (**3**). The items were selected by a method which stressed their intercorrelations. The use of groups of items at each level, and the appearance of a test at two levels with the difference only in the method of scoring might raise the intercorrelations. The common practice in giving the Binet of establishing a basal age below which all tests are regarded as passed and of assuming that all tests above the level of complete failure are failed might also affect the results.

The second hypothesis of maturation seems more tenable on the basis of available experimental evidence. The present study is an analysis of the test results of children who were ten years old, but who ranged in mental age from six to fourteen. Thus there was a wide divergence of both rate and level of mental development in this population. Thurstone, in three separate analyses of test batteries, found no general factor. One study was on a population of college students (**5**). The other two were on high-school students at Lane (**4**) and Hyde Park High Schools (**7**) respectively. Most of the high-school students were seniors. Since from a psychological viewpoint of mental development, these subjects were adults, the absence in all three studies of a common factor, present in a study of children, supports the maturation theory. Although in this study the general factor is tentatively interpreted as an effect of maturation, further studies are necessary to test this hypothesis.

The first column of the rotated factorial matrix, Table 4, has twenty-three entries which are nearly zero. The tests with projections high enough for consideration in an attempt to name the factor are:

4.	Counting backwards from twenty to one	.530
12.	Repeating four digits backwards	.506
11.	Making change	.381
9.	Dates	.337
26.	Repeating five digits backwards	.333

The characteristic common to these variables is apparently the presence of number operations of some sort. Of the items which do not appear in this list, the only one involving numbers to any extent is item 1, repeating five digits. It has a loading of only .244. Possibly this item can be successfully solved by the use of some other ability. It is the easiest of those items involving numbers, and more of its variance is accounted for by the common factor than for any other number test. Both of these considerations support the suggestion that

other abilities may be substituted. The factor is tentatively identified as Number.

The second column has twenty-one nearly vanishing entries. The significant entries are:

17.	Drawing designs from memory	.457
31.	Reversing the hands of a clock	.455
29.	Induction — paper cutting	.429
3.	Ball and field — inferior level	.370
2.	Copying the diamond	.344
23.	Ball and field — superior level	.296

The common element underlying these items seems spatial in character. No loading is particularly high, but evidence of a spatial component is consistent throughout the items. There is no reason to suspect that any item missing from this list has an appreciable amount of space thinking involved. The factor is identified as Space.

The fourth column has twenty-one entries which are nearly zero. The significant loadings are:

21.	Vocabulary — twelve year level	.668
29.	Induction — paper cutting	.371
24.	Dissected sentences	.358
5.	Comprehension — eight year level	.352
22.	Definition of abstract words	.337
25.	Fable interpretation	.316
16.	Verbal absurdities	.292
19.	Comprehension — ten year level	.286

There is little doubt that the element common to these items is concerned with verbal relations. On first inspection, the appearance of item 29 in this group may be puzzling, but if it is remembered that in order to pass this item the method of solution must be given verbal expression, the apparent discrepancy vanishes. There are other items in the scale which one might expect to have some sort of verbal component; however, their failure to appear here is not disconcerting. There may be other abilities dealing with verbal material. Another alternative is that the abilities may be specific to each variable and the variance will therefore be confined to the specific factor space.

The fifth column has twenty-four nearly vanishing entries. The loadings which are high enough for consideration are:

2.	Copying the diamond	.500
17.	Reproducing designs from memory	.428
7.	Definitions better than use	.318
4.	Counting backwards from twenty to one	.304

It does not seem possible to determine the characteristic which is common to these items. The variables have no obvious underlying

similarity. There are too few of them to allow any extensive speculation. Since all the variables lie at the ten year level or below, there is a possibility that the method or ability is more common to children at a lower level of development.

The sixth column has twenty-three entries which approach zero. The significant loadings are:

23.	Ball and field — superior level	.682
3.	Ball and field — inferior level	.605
30.	Problem questions	.341
27.	Interpretation of Pictures	.314

The common feature throughout this list of tests is the application of some sort of reasoning, but the number of variables is not extensive enough to make the identification certain.

Column seven has twenty-two nearly vanishing entries. The largest projections are:

6.	Finding likenesses — two items	.532
16.	Verbal absurdities	.460
27.	Interpretation of pictures	.342
26.	Repeating five digits backwards	.307
28.	Finding likenesses — three items	.301
11.	Making change	.296

The characteristic common to this list of items is tentatively interpreted as induction. The finding of some likeness between items is comparable to finding a rule or principle. A verbal absurdity includes a point which is at variance with the rest of the situation, and requires the detection of a discrepancy from a rule. Interpretation of pictures involves finding a meaning in the picture from all the details. Items 11 and 26 may not, on first glance, be obviously inductive in nature, but it must be remembered that what is inductive at one age level may be a task which requires another ability at a different level. Repeating digits backwards may require the development of some principle of procedure such as repeating the numbers in the forward order first, combining them into groups or devising any other scheme for recall. Making change, for a child, may be a complicated process in which many elements must be taken into consideration, while making change for an adult may be a process of a much simpler nature. Relevant to this problem, Thurstone (**5**, p. 87) states that "another way in which the factorial components of a test may be altered is to give the same test at different levels. A very common example is the case of a simple arithmetic problem, say, $3 \times 1\frac{1}{4}$. To answer a score of such items quickly at the age of fifteen is indicative of the factor

N, but a four-year-old who solves such problems rationally may reveal logical faculties, perhaps inductive."

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STUDIES IN MATHEMATICAL THEORY OF HUMAN RELATIONS

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In continuation of previous studies, an abstract mathematical theory is developed for the interaction of several social classes, of which one influences and controls the behavior of the others. In some cases such interaction results in the existence of two configurations of equilibrium for the social structure, characterized by different types of behavior. Each configuration corresponds to one definite type of behavior. The transition from one configuration to the other, in other words, the transition from one behavior to another, occurs rather rapidly. Equations governing these transitions are given. In other cases, namely when the efforts of the individuals to influence others into a given behavior lessen as the success of this influence increases, there is only one stable configuration characterized by a mixed behavior. Due to the dissimilarity of parents and progeny, the composition of each social class changes with time, as generations change. This results in the appearance of instabilities of the social structure and in relatively sudden social changes. Possibilities of quasi-periodic alterations of different social structures are discussed. Finally, the developed equations are applied to the case of physical conflicts between groups of individuals, such as riots, wars, etc. Possible factors in addition to mere physical force which may determine the outcome of such conflicts are investigated.

I

In two previous publications (**1**, **2**) we developed a general mathematical approach to social phenomena. It has been shown how some general characteristics of sociological changes, leading to instabilities, can be described mathematically. However, the equations involved are of a type that does not lend itself readily to closed solutions, even in simplest cases. As we attempt to describe more complex situations, the mathematical difficulties increase still further. The use of some general approximation method, similar in spirit to the one recently developed in mathematical biophysics (**3**), is therefore indicated.

We have seen that for a given distribution function of some individual characteristic in a given population, this population will split into two or more classes (**1**), due to a tendency of any individual to associate preponderantly with other individuals with a similar characteristic. In the special case when this characteristic is the "coefficient of influence" (**1**) and when the number of classes is two, we have

one class influencing and controlling the activities of the other. The exact interaction of the two classes depends on the structure of the classes, which are not homogeneous with regard to the given characteristic of the individuals constituting the class. This structure is itself determined by the distribution function of the characteristic considered. We obtain a great simplification if we introduce, instead of the actual structure of the classes, an average value of the characteristic of the individuals of a given class. We thus have a class of individuals having a certain characteristic in the amount a_1, another having the same characteristic in the amount a_2, etc. Or, more generally, we may have several classes, characterized by different average amounts of a characteristic a, several other classes characterized by different average amounts of *another* characteristic b, etc. For instance a may represent an average amount of education, while b may represent an average ability to play football. Such classes may or may not overlap. That is, an individual may belong to the class of highly educated people and of average quality football players.

We shall here confine ourselves to the special case when the characteristics of the individuals are their ability to influence activities of other individuals. As we have seen, this ability is itself a composite one. The influence exerted may be due to physical force, to moral persuasion, wealth, etc. We shall consider here at first only one *purely theoretical case*, which presents some interest. Similar considerations may then be applied later on to other different cases.

In actual life, we find that a rather large number of individuals choose their activities not because of any firm conviction of the necessity of such activity, nor because of a particularly strong inclination for this activity, but merely because the majority of the other individuals with whom they come into contact perform this activity. There are of course all gradations from individuals who would never do anything that is not done "by everybody," to those who are guided only by their own convictions and do not care in the least what "people may say about them." Let us first consider, for the sake of simplicity, an abstract, non-existing situation, in which we have two *distinct* groups of individuals, one being composed of individuals who act *only* on their own initiative (type I) and the other being composed of individuals who are strongly influenced by their fellow-individuals and direct their activities exclusively according to what others do (type II). As remarked above, this is a fictitious case, but it may turn out to be a first approximation to the more real case, when there are gradations in the above characteristics. By successive approximations, we may hope to arrive at a treatment of the more concrete cases.

Let us then consider a total number of N individuals forming a given population. Of these N individuals, let a number x_0 belong to the type I, being characterized by a certain activity A. We shall refer to them as individuals I_A. Let y_0 individuals belong also to the type I, but let the activity of their choice be a different one, B, such that A excludes B. We shall refer to them as individuals I_B. We may ask what the behavior of the remaining

$$N' = N - x_0 - y_0 \tag{1}$$

individuals, which we assume to be all of type II, will be. The answer to that question will of course depend on the assumptions which we make concerning the mechanism of influence of some individuals upon others. Let us again, for the sake of simplicity, make one of the simplest possible assumptions, namely, that an individual of the type II is more strongly inclined to behave in a given way the more frequently he comes into contact with individuals of type I, who exhibit this particular behavior. Let each individual of type I_A actively try to influence as many individuals of type II as possible, by actually coming into contact with as many of them as he can. Then, denoting by x the number of individuals of type II, who exhibit the behavior A, because they were influenced to do so by the individuals of the type I_A, we see that x *will increase at a rate* which is approximately proportional to x_0. Let the coefficient of proportionality be a_0, so that the total contribution of the individuals of type I_A to the increase of x is a_0x_0. But the x individuals who now exhibit the behavior A, also come into contact with other individuals of type II who have not yet adopted the behavior A. Therefore, they will also contribute to the rate of increase of x, in the amount ax, a being in general different from a_0. Similarly, denoting by y the number of individuals of type II, who have adopted the behavior B through contact with other individuals, we find that the rate of increase of y is proportional to c_0y_0, c_0 being a coefficient of proportionality, and to ay. a is the same as before, since in both cases the coefficient refers to individuals of type II.

If there is only one alternative for behavior, either A or B, then the increase of x is accompanied by a corresponding decrease of y, and *vice versa*. Therefore we have for the total rate of increase of x:

$$\frac{dx}{dt} = a_0x_0 + ax - c_0y_0 - ay\,; \tag{2}$$

and, correspondingly:

$$\frac{dy}{dt} = c_0y_0 + ay - a_0x_0 - ax\,. \tag{3}$$

Since, by definition of x and y, and because of (1),

$$x + y = N - x_0 - y_0 = N' , \tag{4}$$

the system (2) and (3) reduces actually to one equation, either for x or for y. Let us consider x.

From (4), we have

$$x = N' - y \; ; \quad y = N' - x . \tag{5}$$

Introducing this into (2), we find

$$\frac{dx}{dt} = 2ax - (aN' - a_0x_0 + c_0y_0) . \tag{6}$$

If

$$aN' - a_0x_0 + c_0y_0 < 0 , \tag{7}$$

then for $x = 0$, $dx/dt > 0$, and it remains positive as x increases. In other words, if inequality (7) is satisfied, then, if originally all individuals, except x_0, showed the behavior B, the number of individuals showing behavior A will increase, until all, except y_0, will exhibit behavior A. Introducing for N' the expression (1), we find that inequality (7) is equivalent to

$$x_0 > \frac{a}{a_0 + a} N + \frac{c_0 - a}{a_0 + a} y_0 . \tag{8}$$

By a similar argument, starting with equation (3), we find that if

$$y_0 > \frac{a}{c_0 + a} N + \frac{a_0 - a}{c_0 + a} x_0 \tag{9}$$

then for $y = 0$, $dy/dt > 0$ and remains positive as y increases. When (9) holds, then even if originally all individuals, except y_0, exhibited behavior A, they will all gradually exhibit behavior B, except the number x_0. If (8) is not satisfied, it does not mean that (9) is satisfied, and *vice versa*. Solving (8) with respect to y_0 gives

$$y_0 < \frac{a_0 + a}{c_0 - a} x_0 - \frac{a}{c_0 - a} N \; ; \tag{10}$$

while solving (9) with respect to x_0, gives

$$x_0 < \frac{c_0 + a}{a_0 - a} y_0 - \frac{a}{a_0 - a} N . \tag{11}$$

Thus it may happen that neither (8) nor (9) is satisfied. When this is the case, then for $x = 0$, $dx/dt < 0$; but since x cannot be negative, this simply means that if all individuals, except x_0, exhibit behavior

B, this social configuration is relatively stable, and nothing happens to change it. Similarly, if all individuals except y_0 exhibit behavior A, this is also a stable configuration. The social aggregate thus possesses two stable configurations of equilibrium: one (configuration A), in which all except y_0, exhibit behavior A, another (configuration B) in which all except x_0, exhibit behavior B. In order to bring the social aggregate from configuration B into configuration A, some external disturbance, (e.g., intervention of a different social aggregate, war) must make

$$x > \frac{aN' - a_0 x_0 + c_0 y_0}{2a} = \frac{aN - (a_0 + a) x_0 + (c_0 - a) y_0}{2a}. \tag{12}$$

Then, as is readily seen from equation (6), $dx/dt > 0$, and x will increase until it becomes equal to $N' = N - x_0 - y_0$. Similarly, in order to bring the social aggregate from configuration A into configuration B, an external disturbance must make

$$y > \frac{aN' - c_0 y_0 + a_0 x_0}{2a} = \frac{aN - (c_0 + a) y_0 + (a_0 - a) x_0}{2a}. \tag{13}$$

While neither (8) nor (9) may be satisfied, it cannot happen that they both are satisfied. Thus we have: when (8) is satisfied, the only stable configuration is A; when (9) is satisfied, the only stable configuration is B. When neither (8) nor (9) is satisfied, there are two stable configurations, A and B.

Let us analyse the meaning of the coefficients a_0, c_0, a, and of the inequalities (8) and (9). a_0 is the average number of individuals of type II, with whom every individual of type I_A comes into contact per unit time. c_0 has a similar meaning with respect to individuals I_B, while a is the average number of individuals of type II with whom every individual of type II comes in contact per unit time. All three are definitely measurable quantities. The individuals may come in contact with each other directly, or indirectly, through mail, press or radio. Thus a_0, c_0 and a are influenced by any developments of means of communications, in the broadest meaning of this word. These coefficients may be expressed as functions of the average circulation of mail m, average circulation of newspapers n, number of broadcasting stations b, and number of radio receivers r. Thus

$$\begin{aligned} a_0 &= f_1(m, n, b, r) \; ; \\ c_0 &= f_2(m, n, b, r) \; ; \\ a &= f_3(m, n, b, r) \; . \end{aligned} \tag{14}$$

If the groups I_A and I_B are particularly eager to induce type II to exhibit their respective behaviors, they would endeavor to increase a_0 and c_0 by varying appropriately m, n, b and r as far as they can.

This suggests the investigation of the case in which

$$a_0 \sim c_0 \; ; \quad a_0 >> a \; ; \quad c_0 >> a \, . \tag{15}$$

In this case, the expressions

$$\frac{a}{a_0 + a} \text{ and } \frac{a}{c_0 + a} \tag{16}$$

in (8) and (9) are much smaller than unity, while

$$\frac{c_0 - a}{a_0 + a} \text{ and } \frac{a_0 - a}{c_0 + a} \tag{17}$$

are of the order of magnitude of unity. Both x_0 and y_0 may be rather small fractions of N. If, for instance, (8) is satisfied, so that the society is in configuration A, and then x_0 gradually decreases or y_0 increases, so that finally (9) becomes satisfied, then at that moment y will begin to increase, until the whole society passes into configuration B. If both x_0 and y_0 are small fractions of N, then a relatively small number of individuals added to either I_A or I_B may produce a sudden change in the behavior of the whole society. A numerical example will illustrate this. First let $a_0 = c_0 = 1000$ individuals per day, while $a = 10$ individuals per day. Let N be 10^7 individuals and yet y_0 be 10^5 individuals so that $y_0 = 0.01N$. In this case, inequality (8) gives:

$$x_0 > 1{\cdot}97 \times 10^5 \, . \tag{18}$$

Let $x_0 = 2 \times 10^5$ individuals, so that (18) is satisfied, so that the whole social aggregate except y_0 individuals exhibit behavior A. Let, now, for one reason or another, x_0 decrease to 10^5, while y_0 increases to 2×10^5. Then (18) will cease to be satisfied, while (9) becomes satisfied and the configuration begins to change from A to B. Thus a change of 100000 in a population of 10000000 produces a complete change in the behavior of the whole population.

The dynamics of the transition from configuration A to the configuration B, and inversely, is given by equations (2) and (3). Consider, for instance, the transition from B to A. Putting

$$aN' - a_0 x_0 + c_0 y_0 = - u \, , \tag{19}$$

equation (6) becomes

$$\frac{dx}{dt} = 2ax + u \; ; \tag{20}$$

which, integrated, gives

$$x = \frac{C}{2a} e^{2at} - \frac{u}{2a},$$

C being a constant of integration. If for $t = 0$, $x = 0$, then

$$C = u,$$

and hence

$$x = \frac{u}{2a}(e^{2at} - 1). \tag{21}$$

The configuration A is reached when $x = N - x_0 - y_0$. Hence the moment t_A at which this occurs is determined by

$$\frac{u}{2a}(e^{2at_A} - 1) = N - x_0 - y_0 \tag{22}$$

or

$$t_A = \frac{1}{2a} \log \frac{2a(N - x_0 - y_0) + u}{u}. \tag{23}$$

With the above-used values for a_0, c_0, a and x_0, we find that t_A is of the order of a day. This agrees in general with the rapid spread of all sorts of mass hysterias, revolts, etc.

As we already remarked, the coefficients a_0, c_0 and a are functions of various parameters characterizing technical facilities at the disposal of the individual. With increasing technical facilities, they also increase, thus shortening t_A. These coefficients are also functions of the personal endeavor of the individuals to influence others. In general, we would expect that the element of personal effort will be more strongly pronounced in individuals of type *I* than in those of type *II*. Hence while a will be relatively constant, for constant m,n,b,r (cf. equ. 12), a_0 and c_0 will vary within a rather wide range. This need not necessarily be so. But for definiteness let us consider here this particular case, without any prejudice against the other possibilities. In this case, a_0 and a might well be called "coefficients of propaganda," if it were not for the circumstance that different existing connotations of the word "propaganda" may make such a terminology objectionable.

Let us consider that the efforts of individuals I_A in influencing individuals of type *II* decrease as the number of influenced individuals increases. This is psychologically a rather plausible situation. With increasing success, when its permanence appears assured, some people are apt to decrease their efforts. In this case a_0 will be a decreasing function of x. In the simplest case, it will be a linear function, so that

$$a_0 = a_0^*(1 - \varepsilon x)\ . \tag{24}$$

Similarly we put

$$c_0 = c_0^*(1 - \varepsilon' y\ ). \tag{25}$$

After rearrangement, equation (6) now becomes, because of (5):

$$\frac{dx}{dt} = (2a - a_0^* \varepsilon x_0 - c_0^* \varepsilon' y_0)\, x$$

$$- \left[(a - c_0^* \varepsilon' y_0)\, N' - a_0^* x_0 + c_0^* y_0 \right]. \tag{26}$$

If the expression in brackets is negative, which, because of (1), is equivalent to

$$x_0 > \frac{a - c_0^* \varepsilon' y_0}{a_0^* + a - c_0^* \varepsilon' y_0} N + \frac{c_0^* \varepsilon' y_0 + c_0^* - a}{a_0^* + a - c_0^* \varepsilon' y_0} y_0 = X_1\ ; \tag{27}$$

then for $x = 0$, $dx/dt > 0$, and the influence of the x_0 individuals I_A increases. However, it may happen that while (27) is satisfied, the coefficient of x in equ. (26) may be either negative or positive. If it is negative, we have

$$x_0 > \frac{2a - c_0^* \varepsilon' y_0}{a_0^* \varepsilon} = X_2\,. \tag{28}$$

Putting

$$2a - a_0^* \varepsilon x_0 - c_0^* \varepsilon' y_0 = C_1\ ; \tag{29}$$

$$(a - c_0^* \varepsilon' y_0) N' - a_0^* x_0 + c_0^* y_0 = C_2\ ; \tag{30}$$

equation (26) becomes

$$\frac{dx}{dt} = C_1 x - C_2\,. \tag{31}$$

If both (27) and (28) are satisfied, then $C_1 < 0$; $C_2 < 0$. For $x = 0$, $dx/dt > 0$, but $dx/dt = 0$ for

$$x = \frac{C_2}{C_1} = \frac{a_0^* x_0 - c_0^* y_0 - (a - c_0^* \varepsilon' y_0) N'}{a_0^* \varepsilon x_0 + c_0^* \varepsilon' y_0 - 2a} > 0\,. \tag{32}$$

For $x > C_2/C_1$, $dx/dt < 0$. Hence, in this case, we have a configuration of stable equilibrium, defined by (32), in which a part of the individuals of type II exhibit behavior A, another part exhibit behavior B, This is different from the previously studied case, where in a stable equilibrium either all individuals of type II exhibited behavior A, or all of them exhibited behavior B. The ratio x/y of individuals of type II exhibiting behavior A, to those exhibiting behavior B, shifts continuously in one direction or another as x_0 and y_0 vary continuously.

There are no sudden changes as long as (27) and (28) are satisfied. The equilibrium configuration (32) is reached, starting with any other configuration asymptotically, as is readily seen by integrating (31). This gives

$$x = \frac{C_2}{C_1} + \frac{C_0}{C_1} e^{C_1 t} ,$$

where C_0 is a constant of integration. Since $C_1 < 0$, the second term vanishes with increasing t, and x approaches the values C_2/C_1.

If neither (27) nor (28) is satisfied, then $C_1 > 0$, $C_2 > 0$, and $dx/dt = 0$ for $x = C_2/C_1 > 0$. But $dx/dt > 0$ for $x > C_2/C_1$, and $dx/dt < 0$ for $x < C_2/C_1$. The configuration $x = C_2/C_1$ is unstable, and again either all individuals of type II show behavior A or they all show behavior B.

We may also investigate the more complex case, in which the rate of increase of x is proportional to the products $x_0 y$ and xy. This may look plausible, since as x increases and therefore y decreases, the chances of an individual I_A or of an individual of type II exhibiting behavior A, to get into contact with individuals II_B, decrease.

II

Let us now again consider a population of N individuals, with x_0 individuals I_A and y_0 individuals I_B, the remainder being of type II. Let inequality (8) be satisfied, so that all N' individuals of type II exhibit behavior A. By virtue of associations with similar individuals, all individuals of type I_A will form a social class, in a manner previously described. Since we are dealing with a simplified case of a uniformity of all individuals of a type, this social class will be composed with the same approximation of the x_0 individuals I_A. Similarly there will be a social class of y_0 individuals of type I_B, and a social class of N' individuals II. Using the terminology of our previous paper, we may say that the first social class (I_A) controls the behavior of the whole population. With the majority of the population exhibiting behavior A, the group I_B may either be prevented from activity B, or even entirely suppressed and destroyed.

Let us study now the variation of the composition of the controlling class with respect to time, under the assumptions made previously. Namely, we take into account that of the total progeny of individuals I_A only a small fraction α will itself belong to a type I_A, the remainder belonging to type II. However, let the progeny of the first class associate only with progeny of the same class. That is, instead

of association by actual similarity, we shall have an association by the *similarity of the past generations.* As we have seen elsewhere, this results in an increase of individuals of type II in the first class. We shall call class A the social class composed *initially* of individuals of type I_A, we shall call class B the social class composed initially of individuals of type I_B, while the social class composed initially of individuals of type II will be referred to as class II.

Let n_A be the total number of individuals in the social class A. Let $n_A{}^A$ be the number of individuals in class A of the type I_A, $n_A{}^{II}$ be the number of individuals in class A of type II, and $n_A{}^B$ be the number of individuals in class A of type I_B. Similarly, we denote by n_{II} the total number of individuals in class II, by $n_{II}{}^{II}$ the number of individuals in class II of type II, by $n_{II}{}^A$ the number of individuals in class II of type A, and by $n_{II}{}^B$ the number of individuals in class II of type B. And, in a similar way, we define n_B, $n_B{}^A$, $n_B{}^{II}$, $n_B{}^B$. We have

$$N = n_A + n_{II} + n_B , \tag{33}$$

and

$$n_A = n_A{}^A + n_A{}^{II} + n_A{}^B \; ; \quad n_B = n_B{}^A + n_B{}^{II} + n_B{}^B ,$$
$$n_{II} = n_{II}{}^{II} + n_{II}{}^A + n_{II}{}^B . \tag{34}$$

Consider the case in which the rate of increase of the population is proportional to the population. Then:

$$\frac{dn_A}{dt} = \beta_A n_A \; ; \quad \frac{dn_{II}}{dt} = \beta_{II} n_{II} \; ; \quad \frac{dn_B}{dt} = \beta_B n_B . \tag{35}$$

In general, of course, we should take three different values, β_A, β_{II} and β_B. We consider here, only for sake of simplicity, the case that

$$\beta_A = \beta_{II} = \beta_B = \beta . \tag{36}$$

The net change cannot be considered, strictly speaking, as simply a difference between birth and mortality. Not only are new types born, but the old ones die out. In other words, we should consider also a differential mortality for the different types. For simplicity, we omit here this complication, which will be discussed in a further investigation.

Of all the progeny in any class, let the fraction α_A be of the type I_A, the fraction α_B be of type I_B, the remainder $1 - \alpha_A - \alpha_B$ being of type II. Strictly speaking, these fractions should differ for different classes, but again for simplicity we consider them the same for all classes. The more general case is treated in a similar way. We now have

$$\frac{dn_A{}^A}{dt} = \alpha_A \beta n_A \; ; \quad \frac{dn_{II}{}^A}{dt} = \alpha_A \beta n_{II} \; ; \quad \frac{dn_B{}^B}{dt} = \alpha_B \beta n_B \, , \tag{37}$$

and similar equations for other $n_i{}^k$.

Equation (35) and (36), together with the consideration that initially we have $n_A = x_0$, $n_B = y_0$, $n_{II} = N_0{}'$, give for any value of $t > 0$:

$$n_A = x_0 e^{\beta t} \; ; \quad n_{II} = N_0{}' e^{\beta t} \; ; \quad n_B = y_0 e^{\beta t} \, . \tag{38}$$

Substituting (38) into (37), we have:

$$\frac{dn_A{}^A}{dt} = \alpha_A \beta x_0 e^{\beta t} \, , \quad \frac{dn_{II}{}^A}{dt} = \alpha_A \beta N_0{}' e^{\beta t} \, , \quad \frac{dn_B{}^B}{dt} = \alpha_B \beta y_0 e^{\beta t} \, . \tag{39}$$

with similar equations for other $n_i{}^k$.

Equations (39) give:

$$\begin{gathered} n_A{}^A = \alpha_A x_0 e^{\beta t} + C' \, , \quad n_{II}{}^A = \alpha_A N_0{}' e^{\beta t} + C'' \, , \\ n_B{}^B = \alpha_B y_0 e^{\beta t} + C''' \, , \end{gathered} \tag{40}$$

C', C'' and C''' being integration constants. Since, for $t = 0$,

$$\begin{gathered} n_A{}^A = x_0 \, , \quad n_{II}{}^A = 0 \, , \quad n_B{}^B = y_0 \, , \\ x_0 = \alpha_A x_0 + C' \; ; \quad \alpha_A N_0{}' + C'' = 0 \; ; \quad y_0 = \alpha_B y_0 + C''' \, . \end{gathered} \tag{41}$$

Hence

$$C' = x_0 (1 - \alpha_A) \; ; \quad C'' = - \alpha_A N_0{}' \; ; \quad C''' = y_0 (1 - \alpha_B) \, . \tag{42}$$

Introducing (42) into (40) gives:

$$\begin{gathered} n_A{}^A = x_0 + \alpha_A x_0 (e^{\beta t} - 1) \; ; \quad n_{II}{}^A = \alpha_A N_0{}' (e^{\beta t} - 1) \; ; \\ n_B{}^B = y_0 + \alpha_B y_0 (e^{\beta t} - 1) \, . \end{gathered} \tag{43}$$

Comparison of (43) with (38) shows that for $t = \propto$

$$n_A{}^A = \alpha_A n_A \; ; \quad n_{II}{}^A = \alpha_A n_{II} \; ; \quad n_B{}^B = \alpha_B n_B \, . \tag{44}$$

Expressions (43) hold under the assumption that the principle of "hereditary classification" is enforced rigidly. In general we have a certain amount of "social mobility" (**4**), so that individuals of class *II* may work themselves up into class *I*. Let a fraction η of all individuals of type *A* born per unit time in class *II* pass into class *I* (**1**). Since per unit time altogether $\alpha_A \beta N_0{}' e^{\beta t}$ (cf. equ. 39) such individuals are born, we now have, instead of (39):

$$\frac{dn_A{}^A}{dt} = \alpha_A \beta x_0 e^{\beta t} + \eta \alpha_A \beta N_0' e^{\beta t} \, ;$$
$$\frac{dn_{II}{}^A}{dt} = (1 - \eta) \alpha_A \beta N_0' e^{\beta t} \, . \tag{45}$$

The equations for the other $n_i{}^k$ remain the same as before. Equations (45) give, with the same initial conditions as before:

$$n_A{}^A = x_0 + \alpha_A (x_0 + \eta N_0') (e^{\beta t} - 1) \; ;$$
$$n_{II}{}^A = (1 - \eta) \; \alpha_A N_0' (e^{\beta t} - 1) \, . \tag{46}$$

Now we have, for $t = \infty$, instead of (44):

$$n_A{}^A = (\alpha_A + \frac{\eta N_0'}{x_0}) n_A \; ; \quad n_{II}{}^A = (1 - \eta) \; \alpha_A n_{II} \, , \tag{47}$$

which reduces to (44) for $\eta = 0$.

Let us consider possible effects of such changes in the composition of the different social classes, as described by the above equations for sufficiently small mobility η. At the beginning, that is at $t = 0$, the class A was the controlling class, because it was all composed of individuals of type I_A, and their number satisfied the "controlling inequality" (8). The class A continues to control the rest of the population by a sort of hysteresis, which may be in this case appropriately called tradition, although the relative number of individuals of type I_A, in class A, the only ones that can *actively* control, gradually decreases. On the other hand, class II acquires a certain fraction of individuals of type I_A, whose behavior is controlled again by tradition by those of class A, although they are actually of a controlling type themselves. Moreover there is the class B, which originally was too small, and therefore did not gain control, but whose relative size might have increased.

Let us first consider the case in which the total number of individuals of type I_B remains negligible. This will happen, when α_B is very small. Then the individuals of type I_A, belonging to the class II by accident of birth, will try to gain control over the whole population. They will actually gain control when an inequality, corresponding to (8), will become satisfied. In this inequality we must now substitute $n_{II}{}^A$ for x_0, and $n_A{}^A$ for y_0, since it is those two groups that now dispute control. The control now will not consist in imposing on the population the behavior A, (since it is already imposed); but while individuals I_A of class I will seek to impose obedience to them (behavior A'), individuals I_A of class II will seek to impose obedience

to them (behavior A''). For N in inequality (8), we must introduce the total population at the time t, which because of (33) and (38), equals

$$N(t) = N_0 e^{\beta t} . \tag{48}$$

Letting c_0 refer to individuals of type I_A in class A, a_0 to those of type I_A in class II, and a to those of type II in class II, and introducing the notations

$$\mu_1 = \frac{a}{a_0 + a} ; \quad \mu_2 = \frac{c_0 - a}{a_0 + a} , \tag{49}$$

the condition for the taking over of control by individuals of type I_A in class II becomes:

$$n_{II}{}^A > \mu_1 N(t) + \mu_2 n_A{}^A . \tag{50}$$

Introducing for $n_{II}{}^A$, $n_A{}^A$, and $N(t)$ their values from expressions (46) and (48) we obtain

$$\begin{aligned}(1 - \eta)\alpha_A N_0'(e^{\beta t} - 1) &> \mu_1 N_0 e^{\beta t} \\ &+ \mu_2[x_0 + \alpha_A(x_0 + \eta N_0')(e^{\beta t} - 1)] .\end{aligned} \tag{51}$$

The time t^* when the change in control will happen is obtained by substituting for the inequality sign a sign of equality and solving the resulting equation for t. This gives, after rearrangements:

$$t^* = \frac{1}{\beta} \log \frac{(1 - \eta)\alpha_A N_0' + \mu_2[x_0 - \alpha_A(x_0 + \eta N_0')]}{(1 - \eta)\alpha_A N_0' - \mu_1 N_0 - \mu_2 \alpha_A(x_0 + \eta N_0')} . \tag{52}$$

Since for $t = 0$, (51) does not hold, and both sides of (51) are exponentially increasing, if (51) is to hold for positive values of t, it must hold for $t = \infty$, which gives

$$(1 - \eta)\alpha_A N_0' > \mu_1 N_0 + \mu_2 \alpha_A (x_0 + \eta N_0') . \tag{53}$$

Inequality (53) shows that the denominator of (52) is positive. Therefore, in order that t^* should be real and positive, the numerator must be greater than the denominator. This gives

$$\begin{aligned}(1 - \eta)\alpha_A N_0' &+ \mu_2[x_0 - \alpha_A(x_0 + \eta N_0')] \\ &> (1 - \eta)\alpha_A N_0' - \mu_1 N_0 - \mu_2\alpha_A(x_0 + \eta N_0')\end{aligned} \tag{54}$$

which is equivalent to

$$\mu_1 N_0 + \mu_2 x_0 > 0 \tag{55}$$

which is always satisfied. Hence t^* is always real and positive if (51)

holds at all. η must not be too large, for when $\eta = 1$, or is near unity, practically all individuals of type I_A will be in class A. For $\eta = 1$, (53) cannot be satisfied.

With

$$\mu_1 = 10^{-2} \; ; \quad \mu_2 \infty 1 \; ; \quad x_0 = 3 \times 10^5 \; ;$$

$$\eta = 10^{-2} \; ; \quad N_0 \infty N_0' = 10^7 \; ;$$

$\alpha_A = 3 \times 10^{-2}$ and $\beta = 10^{-2}$ year^{-1} ; we find from (52): $t^* \infty 10^2 - 10^3$ years.

A somewhat different situation will develop when the class B is originally rather large, so that while (8) holds, yet x_0 exceeds the right-hand side of (8) only a little. In this case, as class A weakens due to a relative decrease of $n_A{}^A$, it may happen that the inequality

$$n_B{}^B + n_{II}{}^B > \frac{a}{a_0 + a} N + \frac{c_0 - a}{a_0 + a} n_A{}^A \tag{56}$$

will begin to hold, where a_0 refers to individuals of type I_B in class II and B, while c_0 refers to individuals of type I_A in class A. We obtain an expression similar to (52) for the time when this occurs. In this case, individuals of type I_B gain control of the population, and the general behavior changes from A' to B.

The actual transition either from A' to A'', or from A' to B is described by equations of the type of equ. (6) or of other types studied in the preceding section. Once a new type of behavior is established, two things may happen. Either the principle of "controlling class heredity," in other words of small mobility, is part of the new behavior, or it is not. In the first case, again a "thinning out" of the controlling class will occur, and this will result in a change from the new behavior to the old one, those changes going on periodically, with a period of the order of a few hundred years. It must be emphasized that the control which a class exercises need not necessarily be political. It may be a cultural control. In general these two will likely be correlated.

If, after a change from behavior-pattern A to behavior-pattern B, the principle of class-heredity is not adhered to, the behavior-pattern B may last indefinitely. This however is not necessarily the case. We have seen that, in such a case (cf. equations 44),

$$n_A{}^A + n_{II}{}^A + n_B{}^A = \alpha_A N \; ; \quad n_A{}^B + n_{II}{}^B + n_B{}^B = \alpha_B N \, . \tag{57}$$

If $\alpha_A >> \alpha_B$, then eventually the number of individuals of type I_A will increase sufficiently for them to gain control of the population. In spite of a large α_A, and even due to it, the control by individuals of

type I_A cannot last indefinitely, because due to the principle of class-heredity the group of individuals of type I_A eventually divides itself into A' and A''. But a behavior-pattern which does not adopt class heredity will under these conditions also not exist indefinitely.

We have studied here the case of an exponentially increasing population. The other extreme, that of a stationary population, can also be easily studied. In this case, N remains constant, and n_A, n_{II} and n_B also remain constant. But $n_A{}^A$, $n_{II}{}^A$, etc., vary. One should also investigate the case of interaction of three or more groups of type I.

Periodical fluctuations of influence of two different groups may also occur, in the absence of class heredity, if we consider that a group associates into a class only upon reaching a certain size, and that the mortality in such a *closed* class is greater than in class II, due to inbreeding. But until a class is formed, the mortality of I_A is the same as that of II. Then such a class A will gradually become relatively smaller, until it is overcome by another group. Since, however, class II produces individuals of type I in a ratio α, then if α is sufficiently large, a group I_A will be regenerated, keeping on regenerating until it becomes so large that it forms an exclusive class, which results in an increase of mortality, etc. Compare this with the discussion of the differences of mortalities in different social groups by P. Sorokin (5).

III

In previous sections we discussed the relations which govern the influence of a relatively small group of "leading" individuals upon the behavior of the larger number of other individuals. These considerations may be applied at first *in abstracto* to a rather interesting and important social problem, namely to the physical conflict between two large groups of individuals, each being led by a corresponding smaller group. A large number of social phenomena, ranging from a street riot to a war involving two or several nations, correspond to the abstract picture, which we shall discuss here.

Consider two populations, one composed of M, the other of N individuals. Let most of the individuals belong to type II, which we shall now call "passive type." Let x_M individuals in the first and x_N individuals in the second population be of type I, which we shall call the "active type." Suppose that the active groups of the two populations are in mutual conflict and that they endeavor to influence the other individuals in their population in such a way that they would participate in the conflict. In general, each population may have two groups of active individuals of opposite behavior, a number x, for in-

stance, wishing a conflict, a number y opposing it. The behavior of the whole population will then be determined by the considerations of section I, which led to expressions (8) and (9). For simplicity, we shall consider here the case in which there is no opposing group. We may then use expression (8), in which we make $y_0 = 0$, and, for x_0 and N, substitute x_M and M, or correspondingly x_N and N. Thus both populations will participate as a whole in a conflict, if

$$x_M > \frac{a_1}{a_{01} + a_1} M \; ; \quad x_N > \frac{a_2}{a_{02} + a_2} N . \tag{58}$$

a_1 and a_{01}, and, correspondingly, a_2 and a_{02} have the same meaning as a and a_0 of section I. The subscripts are added to take into account possible differences in the constant for the two populations.

Putting

$$\frac{a_1}{a_{01} + a_1} = r_1 \; ; \quad \frac{a_2}{a_{02} + a_2} = r_2 \; ; \tag{59}$$

we may write (58) thus:

$$x_M > r_1 M \; ; \quad x_N > r_2 N . \tag{60}$$

Let the conflict consist in an actual destruction of the individuals of one population by those of the other. The same considerations, however, will apply to the case when, instead of an actual destruction, we have just a temporary elimination, by putting the individuals "hors de combat." For brevity we shall use the word "destroy."

Let each individual of population 1 destroy k_1^2 individuals of population 2, while each individual of population 2 destroys k_2^2 individuals of population 1. We then have

$$\frac{dM}{dt} = -k_2^2 N \; ;$$
$$\frac{dN}{dt} = -k_1^2 M . \tag{61}$$

We may consider the more general case, that k_1^2 and k_2^2 are different for the active and passive individuals. We shall limit ourselves here to the more restricted case.

Equations (61) integrated by the standard method give:

$$M = \tfrac{1}{2}\left(M_0 - \frac{k_2}{k_1} N_0\right) e^{k_1 k_2 t} + \tfrac{1}{2}\left(M_0 + \frac{k_2}{k_1} N_0\right) e^{-k_1 k_2 t} \; ;$$
$$N = -\tfrac{1}{2}\left(M_0 - \frac{k_2}{k_1} N_0\right) \frac{k_1}{k_2} e^{k_1 k_2 t} + \tfrac{1}{2}\left(M_0 + \frac{k_2}{k_1} N_0\right) \frac{k_1}{k_2} e^{-k_1 k_2 t} \; ; \tag{62}$$

$$k^2 = \sqrt{k_1^2 k_2^2} = + k_1 k_2 \,. \tag{63}$$

N_0 and M_0 are the total number of individuals for $t = 0$.

If $M_0 > \frac{k_2}{k_1} N_0$, then M consists of two positive terms, one increasing, the other decreasing. For $t = 0$, $M = M_0$, the first term increases less rapidly, than the second decreases. M therefore shall first decrease, then increase. But from the first equation (61) it follows that when M reaches a minimum, so that $dM/dt = 0$, we also have $N = 0$. Thus at the moment $t = t_2^*$, at which M has a minimum, $N = 0$. This means that at $t = t_2^*$, the second population becomes completely destroyed and the conflict ceases through a victory of the first. After that moment there is no further conflict, so that M does not actually increase.

If $M_0 < \frac{k_2}{k_1} N_0$, then M becomes zero, while N reaches a minimum.

While the leaders of the two populations will in general participate in the active conflict, they may be exposed to the danger of destruction to a different extent than the passive individuals. If their exposure to danger is the same as that of the passive individuals, we have:

$$\begin{aligned} \frac{dx_M}{dt} &= \frac{x_M}{M}\frac{dM}{dt}\,, \\ \frac{dx_N}{dt} &= \frac{x_N}{N}\frac{dN}{dt}\,, \end{aligned} \tag{64}$$

which integrated gives:

$$\frac{x_M}{M} = \text{Const.}\,; \quad \frac{x_N}{N} = \text{Const.}\,. \tag{65}$$

Under these conditions, if (60) was satisfied at the beginning, it will remain satisfied throughout the conflict and the conflict will therefore continue until one of the populations is completely destroyed, that is, for either $t = t_1^*$ or $t = t_2^*$. These values are obtained correspondingly as the roots of the equations $M = 0$ and $N = 0$, into which we substitute expressions (62) and which we solve for t. If $M_0 < \frac{k_2}{k_1} N_0$, then t_2^* is imaginary, while

$$t_1^* = \frac{1}{2k^2} \log \frac{\frac{k_2}{k_1} N_0 + M_0}{\frac{k_2}{k_1} N_0 - M_0}\,. \tag{66}$$

If $M_0 > \frac{k_2}{k_1} N_0$, then t_1^* is imaginary and

$$t_2^* = \frac{1}{2k^2} \log \frac{M_0 + \frac{k_2}{k_1} N_0}{M_0 - \frac{k_2}{k_1} N_0}. \tag{67}$$

In general, we shall have instead of (64):

$$\frac{dx_M}{dt} = (\frac{x_M}{M})^m \frac{dM}{dt} \tag{68}$$

and

$$\frac{dx_N}{dt} = (\frac{x_N}{N})^n \frac{dN}{dt}, \tag{69}$$

where m and n may be either smaller or greater than one.

Equations (68) and (69) give:

$$\frac{dx_M}{x_M{}^m} = \frac{dM}{M^m} \tag{70}$$

and

$$\frac{dx_N}{x_N{}^n} = \frac{dN}{N^n}. \tag{71}$$

Equations (70) and (71) give:

$$\frac{1}{x_M{}^{m-1}} = \frac{1}{M^{m-1}} + C_1 \tag{72}$$

and

$$\frac{1}{x_N{}^{n-1}} = \frac{1}{N^{n-1}} + C_2. \tag{73}$$

C_1 and C_2 are integration constants determined by the requirements that for $M = M_0$, $x_M = x_{0M}$ and for $N = N_0$, $x_N = x_{0N}$, where x_{0M} and x_{0N} are the initial values of x_M and x_N. This gives:

$$C_1 = \frac{M_0{}^{m-1} - x_{0M}{}^{m-1}}{M_0{}^{m-1}\, x_{0M}{}^{m-1}}; \quad C_2 = \frac{N_0{}^{n-1} - x_{0N}{}^{n-1}}{N_0{}^{n-1}\, x_{0N}{}^{n-1}}. \tag{74}$$

For $m < 1$ and $n < 1$, $C_1 < 0$ and $C_2 < 0$. For $m > 1$ and $n > 1$, $C_1 > 0$ and $C_2 > 0$. Equations (72) and (73) give:

$$\frac{x_M}{M} = \sqrt[m-1]{\frac{1}{1 + C_1 M^{m-1}}}\,. \quad \frac{x_N}{N} = \sqrt[n-1]{\frac{1}{1 + C_2 N^{n-1}}}. \tag{75}$$

For $m < 1$, x_M/M decreases with decreasing M, because then $C_1 < 0$.

A similar thing holds for x_N/N. If $m < 1$, the destruction of the active individuals in the first population goes on more rapidly than that of the passive. As a result, x_M/M decreases and if it becomes so small that $x_M < r_1M$, while for the other population we still have $x_N > r_2N$, then the active group can no more influence the passive individuals and make them continue the fight. The population quits fighting, becomes demoralized, although at this moment we may still have $M \geqq N$. Similar considerations hold for the second population. If, on the contrary, $m > 1$ and $n > 1$, then x_M/M and x_N/N increase and the fight continues until one of the populations is completely destroyed.

For $m < 1$, the "breakdown" of the first population occurs at the moment t_1, at which

$$\frac{x_M}{M} = r_1, \tag{76}$$

which because of (75), gives:

$$M = \sqrt[m-1]{\frac{r_1^{\,1-m} - 1}{C_1}} \tag{77}$$

Introducing into (77) the expression for M from (62), and solving with respect to t, we thus find the moment t_1. In a similar way, we find the moment t_2, at which the second population would break down, if the conflict continued. Depending on whether $t_1 < t_2$ or $t_1 > t_2$, the first or the second population will be defeated in the conflict.

The author is indebted to Prof. Harold Gulliksen for calling his attention to an error in the second section of this article.

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PSYCHOMETRIKA—VOL. 4, NO. 3
SEPTEMBER, 1939

A COMPUTATIONAL SHORT CUT IN DETERMINING SCALE VALUES FOR RANKED ITEMS

In a study of color preferences, Winch,* using the order of merit technique, adopted as a measure of group preference an index equivalent to the average rank assigned by N individuals. He awarded to the specimen in first place in each array a value of 1, to second place specimens a value of 2, and so on, summed the values awarded to each item for the N arrays and used their relative magnitudes as a measure of group preference. These values divided by N would give the average rank of the specimens.

In the computation of scale values from ranked items by the method developed by Thurstone,† the first step is the determination of the frequency with which each item is preferred to every other for the N arrays. In this method it is assumed that ranked data yield comparative judgments and that they can be treated in the same manner as data from paired comparisons. The first specimen in a ranked order is considered preferred to all specimens following it, and similarly for other specimens.

It is apparent that awarding to each specimen a value of 1 for every comparison in which it is preferred to the other specimen, as is done here, is equivalent to giving it a value equal to the number of specimens to which it is "superior" in the ranked array. Since this latter technique, as used by Winch, gives for each specimen its average rank in the N arrays, the data yielded by the first step in Thurstone's method are exactly proportional to average rank values.

Thurstone's method proceeds from the "raw" proportions to a determination of the scale values of the items by various transformations. However, the relative values of the items are not greatly altered, for we find, empirically, that the final scale values correlate very highly with the average rank position values. The actual values of course are not similar.

In two experiments of the author, wherein two experimental groups of forty-one and ninety-eight subjects, respectively, ranked ten geometrical forms in order of preference, the greatest difference between the scale values and the average rank values in the two group orders was 3.5% and the average difference was 2.2%. The correla-

* Winch, W. H. Colour preferences of school children. *Brit. J. Psychol.*, 1909, **3**, 42-65.

† Thurstone, L. L. A law of comparative judgment. *Psychol. Rev.*, 1927, **34**, 273-286.

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tions were practically unity.

Since the average rank values correlate so highly with the scale values derived from the law of comparative judgment, they appear to be useful in experimental work as a short-cut in determining group preferences from ranked material.

EDWARD N. BARNHART
The Cleveland Museum of Art

PSYCHOMETRIKA—VOL. 4, NO. 3
SEPTEMBER, 1939

RULES FOR PREPARATION OF MANUSCRIPTS FOR PSYCHOMETRIKA

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2. If possible, submit three typewritten copies of the manuscript. Failure to adhere to this rule greatly delays our editorial procedure.
3. Use heavy white typewriter paper, size 8-1/2 x 11. Double-space the lines and allow wide margins for editorial work.
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5. Footnotes should be reduced to a minimum. However, if the number of references does not exceed 8 to 10, they may be handled as footnotes, thus making unnecessary a list of references. Formulas in footnotes should be avoided. Footnotes should be indicated by the following sequence of symbols: * (asterisk or star), † (dagger), ‡ (double dagger), § (section mark), ‖ (parallels), ¶ (paragraph mark). All footnotes should be typed at the bottom of the page of text to which they refer.
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10. We have adopted the forms of citation adopted by the Board of Editors of The American Psychological Association. The proper form for a journal reference is as follows:
    4. Guilford, J. P. A study in psychodynamics. *Psychometrika*, 1939, **4**, 1-23.

    The proper form for a book reference is as follows:
    3. Thurstone, L. L. The vectors of mind. Chicago: Univ. Chicago Press, 1935.

    For further information concerning forms of reference and acceptable abbreviations for journals, prospective authors are referred to McGeoch, John A. Forms of citation adopted by the Board of Editors of The American Psychological Association. Psychol. Bull., 1939, **36**, 25-30.
11. **A manuscript which fails to comply with these requirements will be returned to the author for revision.**

TWELFTH INTERNATIONAL CONGRESS OF PSYCHOLOGY

Preliminary Notice

The Twelfth International Congress of Psychology will be held in Edinburgh, Scotland, from July 22 to July 27, 1940.

The Congress fee has been fixed at thirty shillings (£1:10s.) sterling for active members and fifteen shillings (15s.) sterling for associates.

Arrangements are being made by which a large proportion of the members can be accommodated in University hostels. The inclusive charge for such accommodation will be about fifty shillings (£2:10s.), excluding midday lunches, arrangements for which will be made elsewhere. Members who do not desire accommodation in the hostels can easily secure accommodation in hotels and private hotels, of which Edinburgh has a large number.

When the Committee of Organization is fully constituted, formal invitations will be issued. It will, however, greatly facilitate arrangements if as many as possible will let the General Secretary know now that their attendance at the Congress is probable. In the meantime subjects for symposia, general discussion and lectures are under consideration by a Preliminary Arrangements Committee.

A volume of Proceedings of the Congress, containing abstracts of papers read, will be published; the cost of this volume is included in the Congress fee for active members.

JAMES DREVER,
President of Committee of Organization.
GODFREY THOMSON,
General Secretary.
(Moray House, Edinburgh 8).